# Efficient arithmetic on elliptic curves

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Classic question about the
Diffie-Hellman system:
How quickly can we compute
$n$th powers mod $p$?

Assume that someone gives you $p$;
e.g. $p = 2^{262} - 5081$.

This talk asks
the analogous question
for elliptic-curve Diffie-Hellman:
How quickly can we compute
$n$th multiples in an
elliptic-curve group?

"Elliptic-curve
scalar multiplication."

Assume that someone gives you a field and an elliptic curve.

e.g. NIST P-224: the elliptic curve $y^2 = x^3 - 3x + a_6$ over $\mathbf{Z}/p$.
Here $p = 2^{224} - 2^{96} + 1$
and $a_6 = 18958286285566608$
$00040866854449392$
$64155046809686793$
$2107578723467256 4.$

e.g. NIST P-256.

e.g. Curve25519.

Your task: Given $(x, y)$ on curve,
and given integer $n \geq 0$,
compute $n$th multiple of $(x, y)$
in the elliptic-curve group.

Warning: Answer is *not* $(nx, ny)$
unless you're extremely lucky.
Elliptic-curve point addition
is not vector addition;
$(x, y) + (x', y')$ is almost never
$(x + x', y + y')$.

Can emphasize this by changing
notation: $+$, $\oplus$, $[n]$, etc. But
this talk uses simplified notation.

# Multiples via additions

Typical recursive formulas:

$2P = P+P$. $3P = 2P+P$.

$4P = 2P+2P$. $5P = 3P+2P$.

$6P = 3P+3P$. $7P = 5P+2P$.

$2nP = 7P+(n-7)P$ if $4 \leq n < 8$.

$(2n+1)P = 2nP+P$ if $4 \leq n < 8$.

$(4n+1)P = 4nP+P$ if $4 \leq n < 8$.

$(4n+3)P = 4nP+3P$ if $4 \leq n < 8$.

$2nP = nP+nP$ if $8 \leq n$.

$(8n+1)P = 8nP+P$ if $4 \leq n$.

$(8n+3)P = 8nP+3P$ if $4 \leq n$.

$(8n+5)P = 8nP+5P$ if $4 \leq n$.

$(8n+7)P = 8nP+7P$ if $4 \leq n$.

This "addition chain"
("length-3 sliding windows")
uses $\approx \lg n$ doublings and
$\approx 0.25 \lg n$ more additions
to compute $nP$ for average $n$.

e.g. $\approx 320$ additions for
average $n \in \{0, 1, \ldots, 2^{256} - 1\}$.

Some easy improvements from
fast negation on elliptic curves:
$(16n - 7)P = 16nP - 7P$, etc.
Also use "endomorphisms" for
"Koblitz curves," "GLV curves."

More complicated methods
replace 0.25 by $\approx 1/\lg \lg n$.

# Explicit doubling formulas

On curve $y^2 = x^3 - 3x + a_6$:

$2(x, y) = (x'', y'')$ where
$\lambda = (3x^2 - 3)/2y$,
$x'' = \lambda^2 - 2x$,
$y'' = \lambda(x - x'') - y$.

7 subs etc., 2 squarings,
1 more mult, 1 division.

How do we divide efficiently
in a finite field?

$f/g = fg^{p-2}$ in prime field $\mathbf{Z}/p$.
Can compute $g^{p-2}$ with
$\approx \lg p$ squarings and
$\approx (\lg p)/\lg\lg p$ more mults.

e.g. $p = 2^{224} - 2^{96} + 1$:
223 squarings, 11 more mults.

More generally, $f/g = fg^{q-2}$
in any field of size $q$.

There are faster division methods
(e.g. "Euclid"—beware timing
attacks!); smaller "I/M ratio."
Special methods for some fields.

## Speedup: delay divisions

Division costs many mults
even with fastest division methods.

Save time by delaying divisions.

Naive division-delay method:
Store field elements as fractions
until end of computation.
Divide once before output.

Mult fractions with 2 field mults.
Divide fractions with 2 field mults.
Add fractions with 3 field mults.

## Speedup: unify denominators

For elliptic-curve doubling,
have denominator $2y$
in $\lambda = (3x^2 - 3)/2y$;
denominator $(2y)^2$
in $x'' = \lambda^2 - 2x$;
denominator $(2y)^3$
in $y'' = \lambda(x - x'') - y$.

Subsequent computations will perform separate computations on the denominators $(2y)^2, (2y)^3$ of $x'', y''$.

Save time by manipulating denominators together.

"Jacobian coordinates":

Store $(x, y, z)$ to represent elliptic-curve point $(x/z^2, y/z^3)$.

$2(x/z^2, y/z^3) = (x'', y'')$ where

$\lambda = (3(x/z^2)^2 - 3)/2(y/z^3)$

$= \alpha/2yz$ with $\alpha = 3x^2 - 3z^4$;

$x'' = \lambda^2 - 2(x/z^2)$

$= (\alpha^2 - 8xy^2)/(2yz)^2$;

$y'' = \lambda((x/z^2) - x'') - (y/z^3)$

$= (12xy^2\alpha - \alpha^3 - 8y^4)/(2yz)^3$.

$2(x/z^2, y/z^3) = (x_2/z_2^2, y_2/z_2^3)$
where $z_2 = 2yz$,
$\alpha = 3x^2 - 3z^4$,
$x_2 = \alpha^2 - 8xy^2$,
$y_2 = \alpha(4xy^2 - x_2) - 8y^4$.

Easily compute with 6 squarings,
3 more mults: $x^2$, $z^2$, $z^4$, $y^2$, $y^4$,
$yz$, $xy^2$, $\alpha^2$, $\alpha(\cdots)$.
Also some subs, doublings, etc.

Use fast field arithmetic:
e.g., can delay carries and
reductions in computing $y_2$.

Speedup: difference of squares

Can compute $3x^2 - 3z^4$ as
$3(x - z^2)(x + z^2)$.

Replace 3 squarings by 1 mult,
1 squaring. Revised total:
4 squarings, 4 more mults.

Note:
$3x^2 - 3z^4$ came from $3x^2 - 3$,
derivative of $x^3 - 3x + a_6$.
Wouldn't have same speedup
for, e.g., $x^3 - 5x + a_6$.

Speedup: $f^2, g^2, 2fg$

After computing $f^2$ and $g^2$
can compute $2fg$
as $(f+g)^2 - f^2 - g^2$.

In particular:
After computing $y^2$ and $z^2$
can compute $2yz$
as $(y+z)^2 - y^2 - z^2$.

Replace 1 mult with 1 squaring.
Revised total: 5 squarings,
3 more mults.

# Explicit addition formulas

Similar speedups in formulas
for adding distinct points.

5 squarings, 11 more mults.

Again some opportunities
to delay carries, etc.

# Speedup: cache results

In adding $(x_1/z_1^2, y_1/z_1^3)$
to $(x_2/z_2^2, y_2/z_2^3)$,
compute many intermediates,
including $z_1^2, z_1^3$.

Often add same point again
to a different point;
can reuse $z_1^2, z_1^3$.

"Chudnovsky coordinates."

## Speedup: delay fewer divisions?

Faster divisions sometimes justify delaying fewer divisions.

e.g. Do we really need fractions for $P, 3P, 5P, 7P$?

Can convert $P, 3P, 5P, 7P$ out of Jacobian coordinates with one division, several mults. Then save mults in every addition of $P, 3P, 5P, 7P$. "Mixed coordinates."

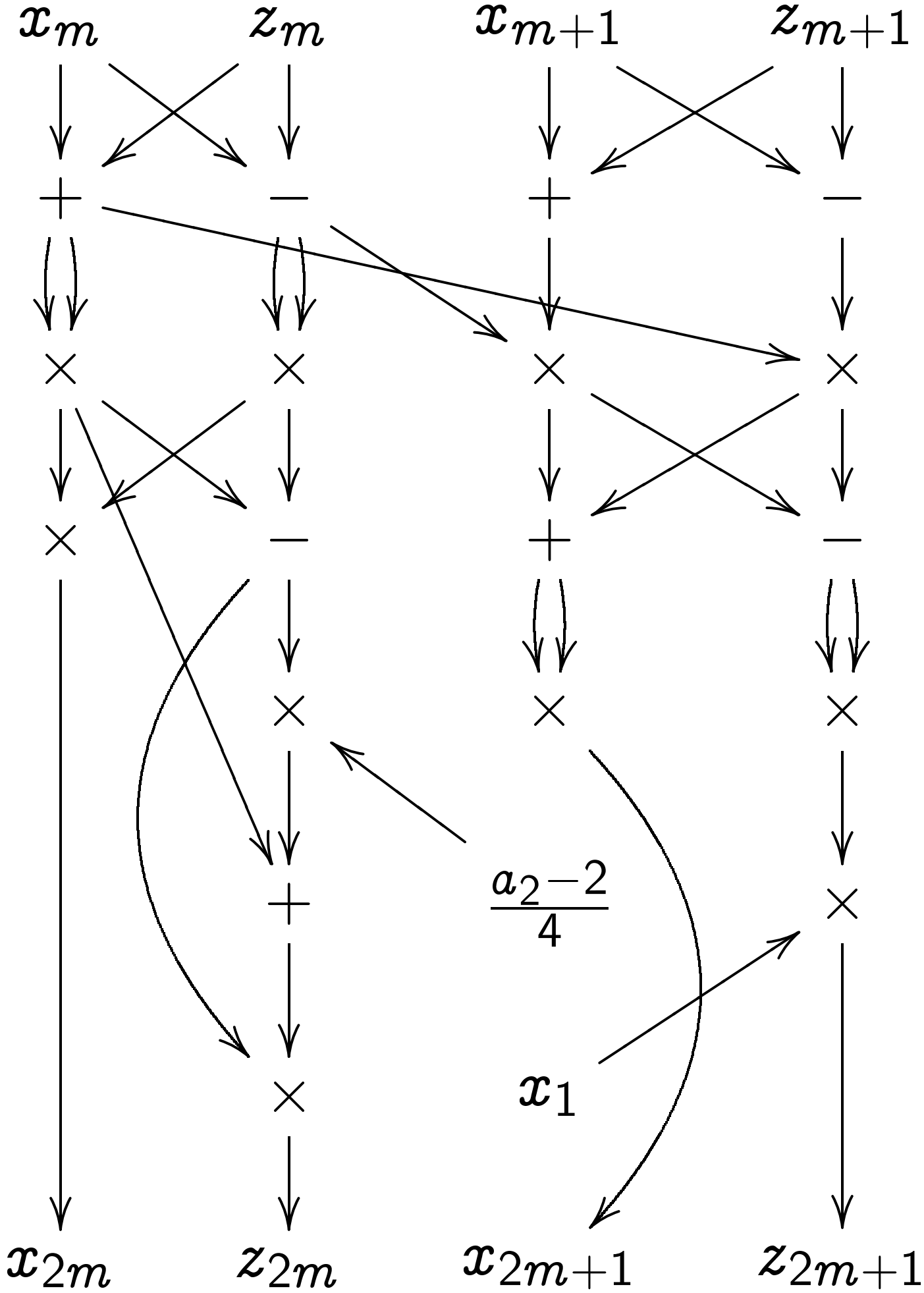
Sometimes worthwhile, depending on division speed.

# Montgomery coordinates

On elliptic curves with
"Montgomery form"
$y^2 = x^3 + a_2x^2 + x$,
preferably with small $(a_2 - 2)/4$:

$n(x_1, \ldots) = (x_n/z_n, \ldots)$ where
$z_1 = 1$; $x_{2m} = (x_m^2 - z_m^2)^2$;
$z_{2m} = 4x_mz_m(x_m^2 + a_2x_mz_m + z_m^2)$;
$x_{2m+1} = 4(x_mx_{m+1} - z_mz_{m+1})^2$;
$z_{2m+1} = 4(x_mz_{m+1} - z_mx_{m+1})^2x_1$.

Can also figure out $y$,
or use cryptographic protocols
that ignore $y$.


$x_m$
$z_m$
$x_{m+1}$
$z_{m+1}$
$\frac{a_2-2}{4}$
$x_1$
$x_{2m}$
$z_{2m}$
$x_{2m+1}$
$z_{2m+1}$

Assuming $(a_2 - 2)/4$ small,
main operations are
4 squarings, 5 more mults
for each bit of $n$.

Compare to Jacobian coordinates:
each bit of $n$ has
5 squarings, 3 more mults,
*and* on occasion
5 more squarings, 11 more mults.

Montgomery form is better
if $n$ is not gigantic.