Efficient arithmetic on elliptic curves

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Classic question about the
Diffie-Hellman system:
How quickly can we compute
nth powers mod p?

"Modular exponentiation."

Assume standard prime p;
e.g. $p = 2^{262} - 5081$.
How quickly can we compute
$g^n \bmod 2^{262} - 5081$,
given integers g, n?

This talk asks
the analogous question
for elliptic-curve Diffie-Hellman:
How quickly can we compute
nth multiples in an
elliptic-curve group?

"Elliptic-curve
scalar multiplication."

Assume standard field
and standard elliptic curve.

e.g. NIST P-224: the elliptic curve $y^2 = x^3 - 3x + a_6$ over $\mathbf{Z}/p$.

Here $p = 2^{224} - 2^{96} + 1$

and $a_6 = $ 18958286285566608
000408668544493926415504680968679321075787234672564.

e.g. NIST P-256: the elliptic curve $y^2 = x^3 - 3x + \cdots$ over $\mathbf{Z}/p$ where $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.

e.g. Curve25519: the elliptic curve $y^2 = x^3 + 486662x^2 + x$ over $\mathbf{Z}/p$ where $p = 2^{255} - 19$.

Your task: Given (x, y) on curve, and given integer $n \geq 0$, compute nth multiple of (x, y) in the elliptic-curve group.

Warning: Answer is *not* (nx, ny) unless you're extremely lucky. Elliptic-curve point addition is not vector addition; $(x, y) + (x', y')$ is almost never $(x + x', y + y')$.

Can emphasize this by changing notation: $+$, $\oplus$, $[n]$, etc. But this talk uses simplified notation.

Similar tasks are critical
for elliptic-curve signatures.

e.g. Schnorr signatures,
unfortunately patented:

Signer has secret key n,
public key nB.

To sign m: choose random z,
uniform in $\{0, 1, \ldots, \#\langle B\rangle - 1$;
compute $r = \text{SHA-256}(zB, m)$;
compute $s = z + rn \bmod \#\langle B\rangle$;
send (m, r, s).

To verify (m, r, s): Check
$r = \text{SHA-256}(sB - rnB, m)$.

Multiples via additions

Typical recursive formulas:

$2P = P+P$. $3P = 2P+P$.

$4P = 2P+2P$. $5P = 3P+2P$.

$6P = 3P+3P$. $7P = 5P+2P$.

$2nP = 7P+(n-7)P$ if $4 \le n < 8$.

$(2n+1)P = 2nP+P$ if $4 \le n < 8$.

$(4n+1)P = 4nP+P$ if $4 \le n < 8$.

$(4n+3)P = 4nP+3P$ if $4 \le n < 8$.

$2nP = nP+nP$ if $8 \le n$.

$(8n+1)P = 8nP+P$ if $4 \le n$.

$(8n+3)P = 8nP+3P$ if $4 \le n$.

$(8n+5)P = 8nP+5P$ if $4 \le n$.

$(8n+7)P = 8nP+7P$ if $4 \le n$.

This "addition chain" ("length-3 sliding windows") uses $\approx \lg n$ doublings and $\approx 0.25 \lg n$ more additions to compute nP for average n.

e.g. ≈ 320 additions for average $n \in \{0, 1, \ldots, 2^{256} - 1\}$.

Some easy improvements from fast negation on elliptic curves: $(16n - 7)P = 16nP - 7P$, etc. Also use "endomorphisms" for "Koblitz curves," "GLV curves."

More complicated methods replace 0.25 by $\approx 1/\lg \lg n$.

Explicit doubling formulas

On curve $y^2 = x^3 - 3x + a_6$:

$2(x, y) = (x'', y'')$ where
$\lambda = (3x^2 - 3)/2y$,
$x'' = \lambda^2 - 2x$,
$y'' = \lambda(x - x'') - y$.

7 subs etc., 2 squarings,
1 more mult, 1 division.

How do we divide efficiently
in a finite field?

$f/g = fg^{p-2}$ in prime field $\mathbf{Z}/p$.
Can compute g^{p-2} with
$\approx \lg p$ squarings and
$\approx (\lg p)/\lg\lg p$ more mults.

e.g. $p = 2^{224} - 2^{96} + 1$:
223 squarings, 11 more mults.

More generally, $f/g = fg^{q-2}$
in any field of size q.

There are faster division methods
(e.g. "Euclid"—beware timing
attacks!); smaller "I/M ratio."
Special methods for some fields.

Speedup: delay divisions

Division costs many mults
even with fastest division methods.

Save time by delaying divisions.

Naive division-delay method:
Store field elements as fractions
until end of computation.
Divide once before output.

Mult fractions with 2 field mults.
Divide fractions with 2 field mults.
Add fractions with 3 field mults.

Speedup: unify denominators

For elliptic-curve doubling,
have denominator $2y$
in $\lambda = (3x^2 - 3)/2y$;
denominator $(2y)^2$
in $x'' = \lambda^2 - 2x$;
denominator $(2y)^3$
in $y'' = \lambda(x - x'') - y$.

Subsequent computations will perform separate computations on the denominators $(2y)^2, (2y)^3$ of x'', y''.

Save time by manipulating denominators together.

"Jacobian coordinates":
Store (x, y, z) to represent elliptic-curve point $(x/z^2, y/z^3)$.

$2(x/z^2, y/z^3) = (x'', y'')$ where
$\lambda = (3(x/z^2)^2 - 3)/2(y/z^3)$
$\quad = \alpha/2yz$ with $\alpha = 3x^2 - 3z^4$;
$x'' = \lambda^2 - 2(x/z^2)$
$\quad = (\alpha^2 - 8xy^2)/(2yz)^2$;
$y'' = \lambda((x/z^2) - x'') - (y/z^3)$
$\quad = (12xy^2\alpha - \alpha^3 - 8y^4)/(2yz)^3$.

$2(x/z^2, y/z^3) = (x_2/z_2^2, y_2/z_2^3)$
where $z_2 = 2yz$,
$\alpha = 3x^2 - 3z^4$,
$x_2 = \alpha^2 - 8xy^2$,
$y_2 = \alpha(4xy^2 - x_2) - 8y^4$.

Easily compute with 6 squarings,
3 more mults: x^2, z^2, z^4, y^2, y^4,
yz, xy^2, α^2, $\alpha(\cdots)$.
Also some subs, doublings, etc.

Use fast field arithmetic:
e.g., can delay carries and
reductions in computing y_2.

Speedup: difference of squares

Can compute $3x^2 - 3z^4$ as
$3(x - z^2)(x + z^2)$.

Replace 3 squarings by 1 mult,
1 squaring. Revised total:
4 squarings, 4 more mults.

Note:
$3x^2 - 3z^4$ came from $3x^2 - 3$,
derivative of $x^3 - 3x + a_6$.
Wouldn't have same speedup
for, e.g., $x^3 - 5x + a_6$.

Speedup: $f^2, g^2, 2fg$

After computing f^2 and g^2
can compute $2fg$
as $(f+g)^2 - f^2 - g^2$.

In particular:
After computing y^2 and z^2
can compute $2yz$
as $(y+z)^2 - y^2 - z^2$.

Replace 1 mult with 1 squaring.
Revised total: 5 squarings,
3 more mults.

Explicit addition formulas

Similar speedups in formulas
for adding distinct points.

5 squarings, 11 more mults.

Again some opportunities
to delay carries, etc.

Speedup: cache results

In adding $(x_1/z_1^2, y_1/z_1^3)$
to $(x_2/z_2^2, y_2/z_2^3)$,
compute many intermediates,
including z_1^2, z_1^3.

Often add same point again
to a different point;
can reuse z_1^2, z_1^3.

"Chudnovsky coordinates."

Speedup: delay fewer divisions?

Faster divisions sometimes justify delaying fewer divisions.

e.g. Do we really need fractions for $P, 3P, 5P, 7P$?

Can convert $P, 3P, 5P, 7P$ out of Jacobian coordinates with one division, several mults. Then save mults in every addition of $P, 3P, 5P, 7P$. "Mixed coordinates."

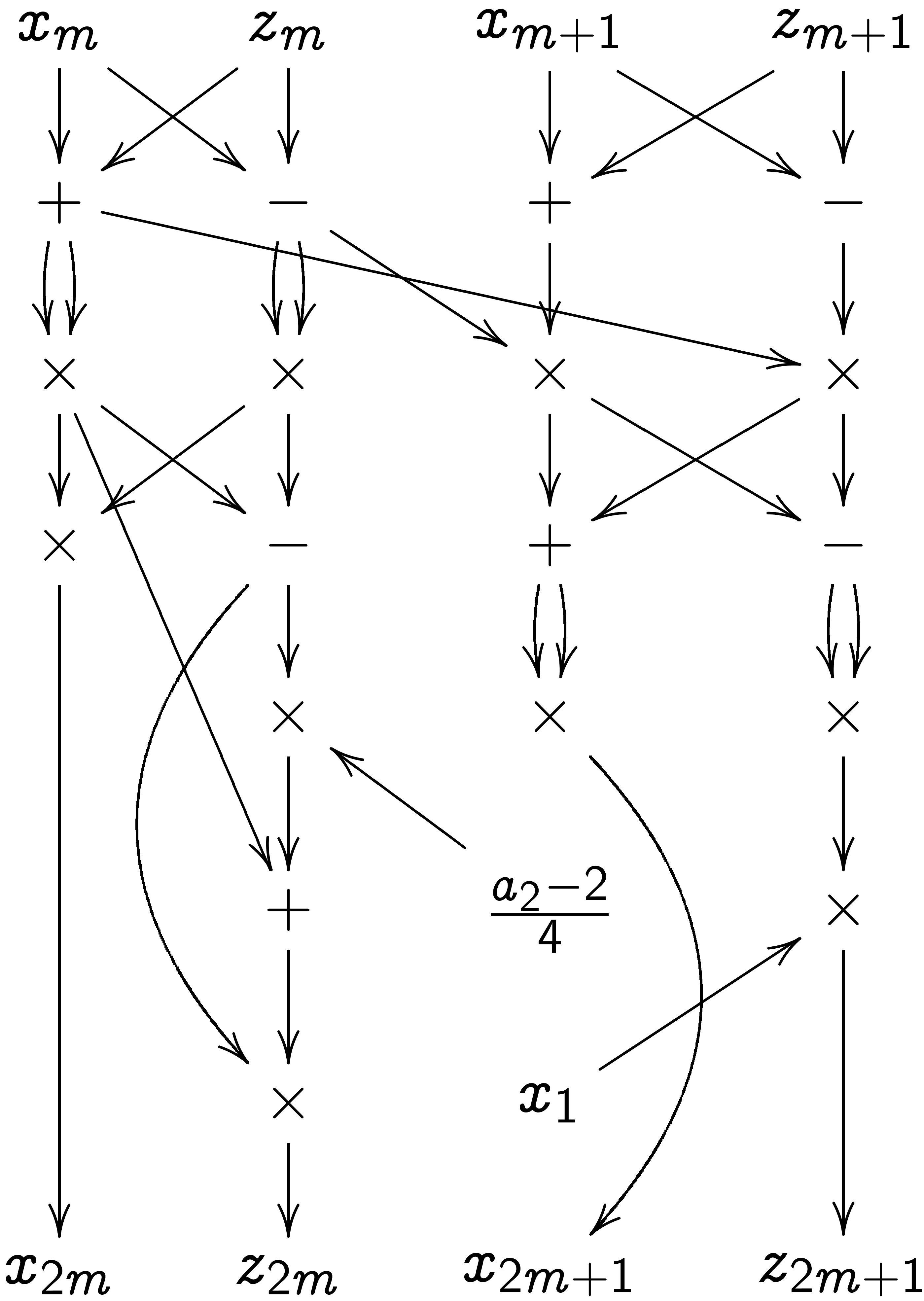
Sometimes worthwhile, depending on division speed.

Montgomery coordinates

On elliptic curves with "Montgomery form" $y^2 = x^3 + a_2x^2 + x$, preferably with small $(a_2 - 2)/4$:

$n(x_1, \ldots) = (x_n/z_n, \ldots)$ where
$z_1 = 1$; $x_{2m} = (x_m^2 - z_m^2)^2$;
$z_{2m} = 4x_m z_m(x_m^2 + a_2 x_m z_m + z_m^2)$;
$x_{2m+1} = 4(x_m x_{m+1} - z_m z_{m+1})^2$;
$z_{2m+1} = 4(x_m z_{m+1} - z_m x_{m+1})^2 x_1$.

Can also figure out y, or use cryptographic protocols that ignore y.



x_m
z_m
x_{m+1}
z_{m+1}
$\frac{a_2-2}{4}$
x_1
x_{2m}
z_{2m}
x_{2m+1}
z_{2m+1}

Assuming $(a_2 - 2)/4$ small,
main operations are
4 squarings, 5 more mults
for each bit of n.

Compare to Jacobian coordinates:
each bit of n has
5 squarings, 3 more mults,
and on occasion
5 more squarings, 11 more mults.

Montgomery form is better
if n is not gigantic.