Elliptic curves over $\mathbf{R}$ and $\mathbf{F}_q$

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Why elliptic curves?

Can quickly compute
$4^n \bmod 2^{262} - 5081$
given $n \in \{0, 1, 2, \ldots, 2^{256} - 1\}$.

Similarly, can quickly compute
$4^{mn} \bmod 2^{262} - 5081$
given n and $4^m \bmod 2^{262} - 5081$.

"Discrete-logarithm problem":
given $4^n \bmod 2^{262} - 5081$, find n.
Is this easy to solve?

Diffie-Hellman secret-sharing system using $p = 2^{262} - 5081$:

Alice's secret key m

Bob's secret key n

Alice's public key $4^m \bmod p$

Bob's public key $4^n \bmod p$

{Alice, Bob}'s shared secret $4^{mn} \bmod p$ = {Bob, Alice}'s shared secret $4^{mn} \bmod p$

Can attacker find $4^{mn} \bmod p$?

Bad news: DLP can be solved at surprising speed! Attacker can find m and n by "index calculus."

To protect against this attack, replace $2^{262} - 5081$ with a much larger prime. *Much* slower arithmetic.

Alternative: Elliptic-curve cryptography. Replace $\{1, 2, \ldots, 2^{262} - 5082\}$ with a comparable-size "safe elliptic-curve group." *Somewhat* slower arithmetic.

An elliptic curve over **R**

Consider all pairs
of real numbers x, y
such that $y^2 - 5xy = x^3 - 7$.

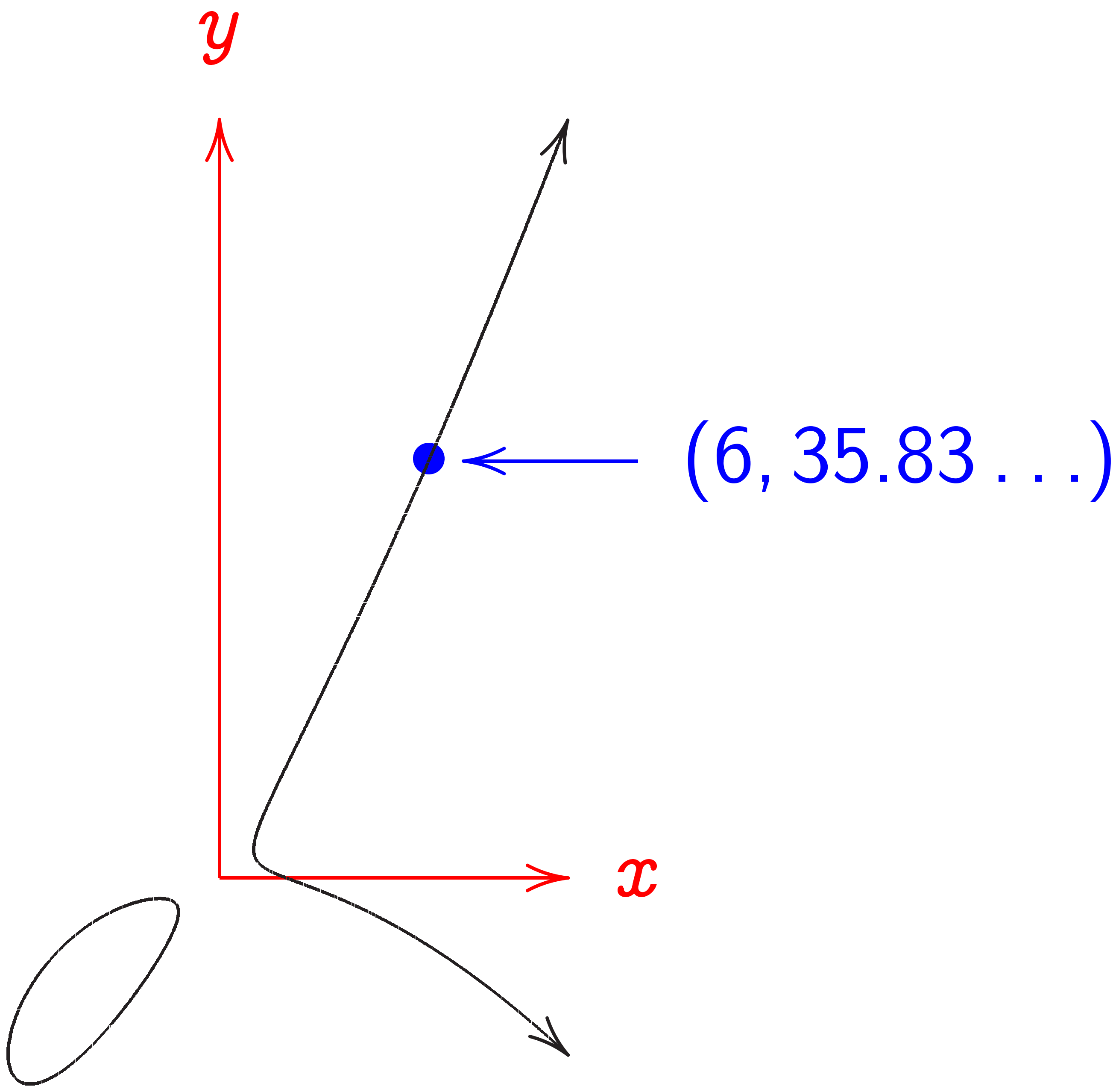
The "points on the elliptic curve
$y^2 - 5xy = x^3 - 7$ over **R**"
are those pairs and
one additional point, ∞.

i.e. The set of points is

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : \\ y^2 - 5xy = x^3 - 7 \ \cup \{\infty \ .$$

(**R** is the set of real numbers.)

Graph of this set of points:



Don't forget ∞.

Visualize ∞ as top of y axis.

There is a standard definition of $0, -, +$ on this set of points.

Magical fact: The set of points is a "commutative group"; i.e., these operations $0, -, +$ satisfy every identity satisfied by $\mathbf{Z}$.

e.g. All $P, Q, R \in \mathbf{Z}$ satisfy $(P + Q) + R = P + (Q + R)$, so all curve points P, Q, R satisfy $(P + Q) + R = P + (Q + R)$.

($\mathbf{Z}$ is the set of integers.)

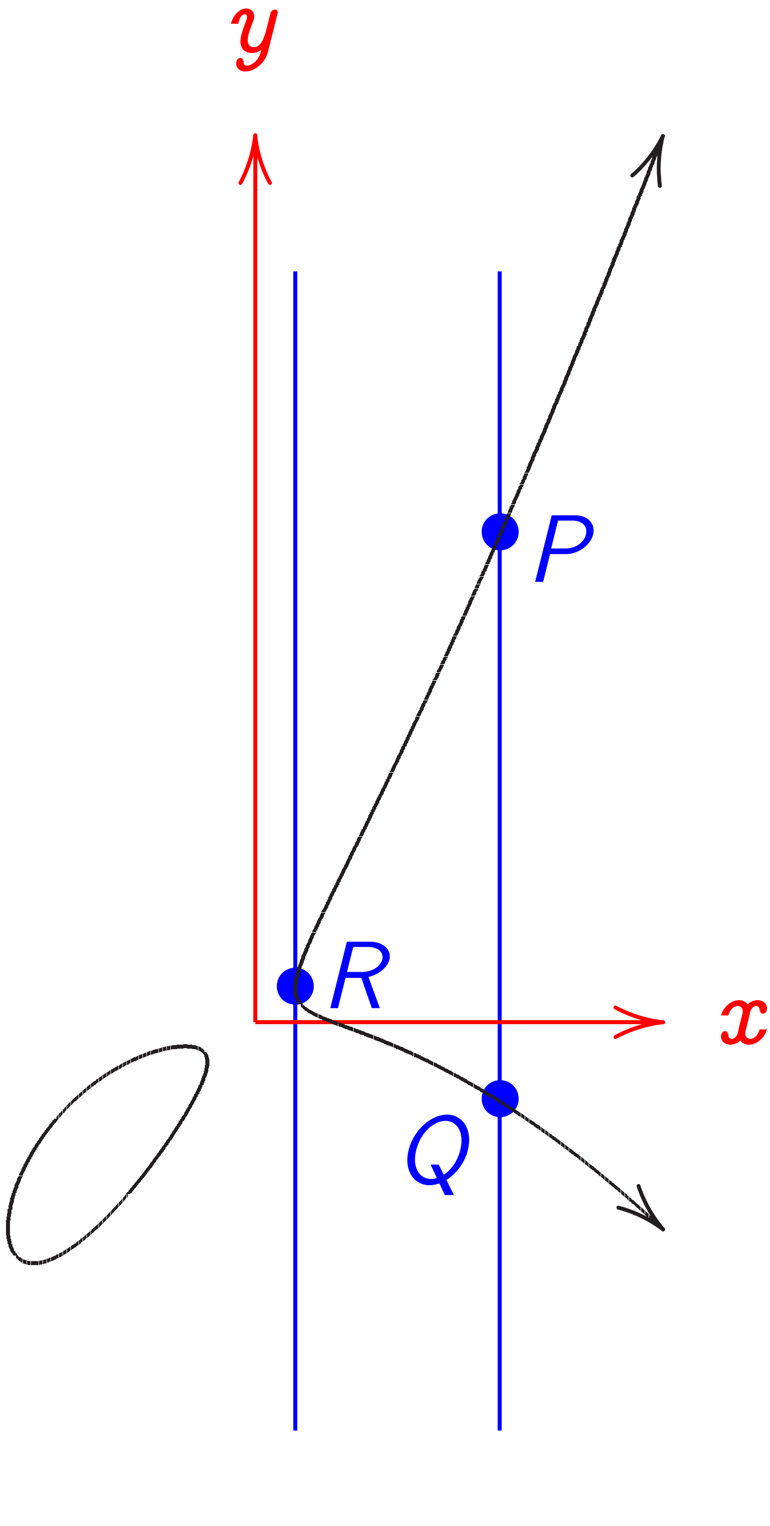
Visualizing the group law

$0 = \infty$; $-\infty = \infty$.

Distinct curve points P, Q
on a vertical line
have $-P = Q$;
$P + Q = 0 = \infty$.

A curve point R
with a vertical tangent line
has $-R = R$;
$R + R = 0 = \infty$.

$-P = Q, \; -Q = P, \; -R = R$:

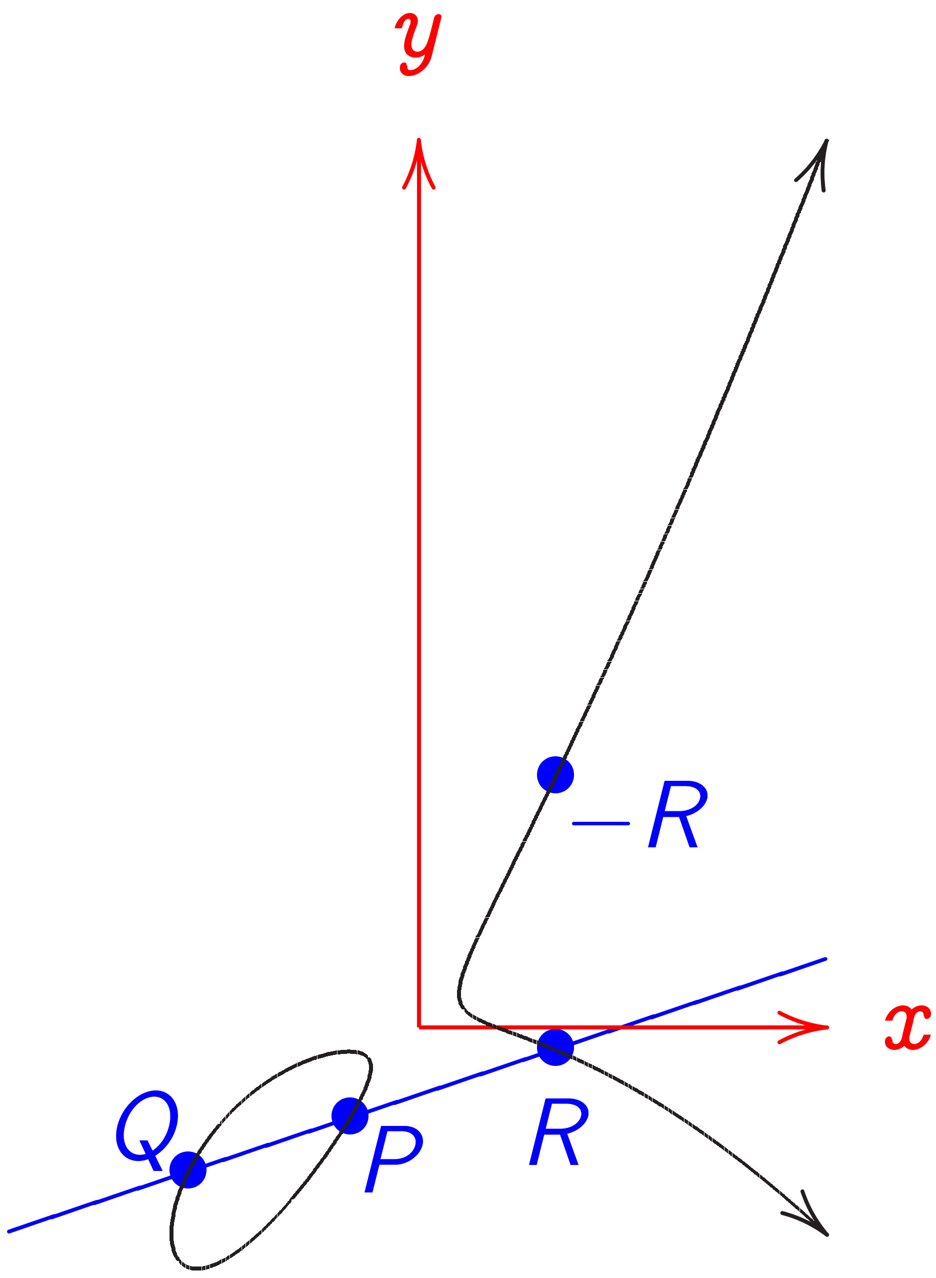


Distinct curve points P, Q, R
on a line
have $P + Q = -R$;
$P + Q + R = 0 = \infty$.

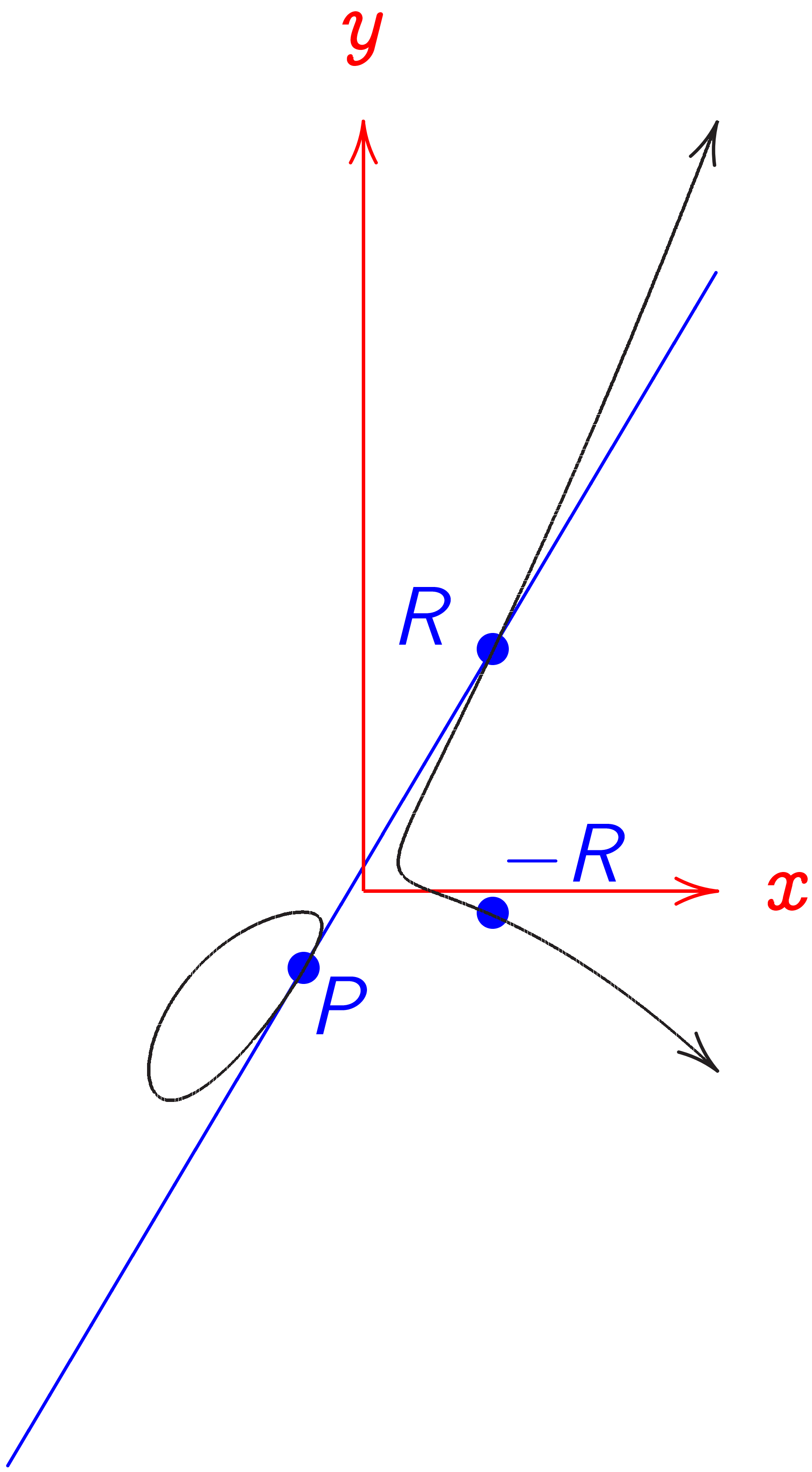
Distinct curve points P, R
on a line tangent at P
have $P + P = -R$;
$P + P + R = 0 = \infty$.

A non-vertical line
with only one curve point P
has $P + P = -P$;
$P + P + P = 0$.

$P + Q = -R$:



$P + P = -R$:



Curve addition formulas

Easily find formulas for $+$
by finding formulas for lines
and for curve-line intersections.

$x \neq x'$: $(x, y) + (x', y') = (x'', y'')$
where $\lambda = (y' - y)/(x' - x)$,
$x'' = \lambda^2 - 5\lambda - x - x'$,
$y'' = 5x'' - (y + \lambda(x'' - x))$.

$2y \neq 5x$: $(x, y) + (x, y) = (x'', y'')$
where $\lambda = (5y + 3x^2)/(2y - 5x)$,
$x'' = \lambda^2 - 5\lambda - 2x$,
$y'' = 5x'' - (y + \lambda(x'' - x))$.

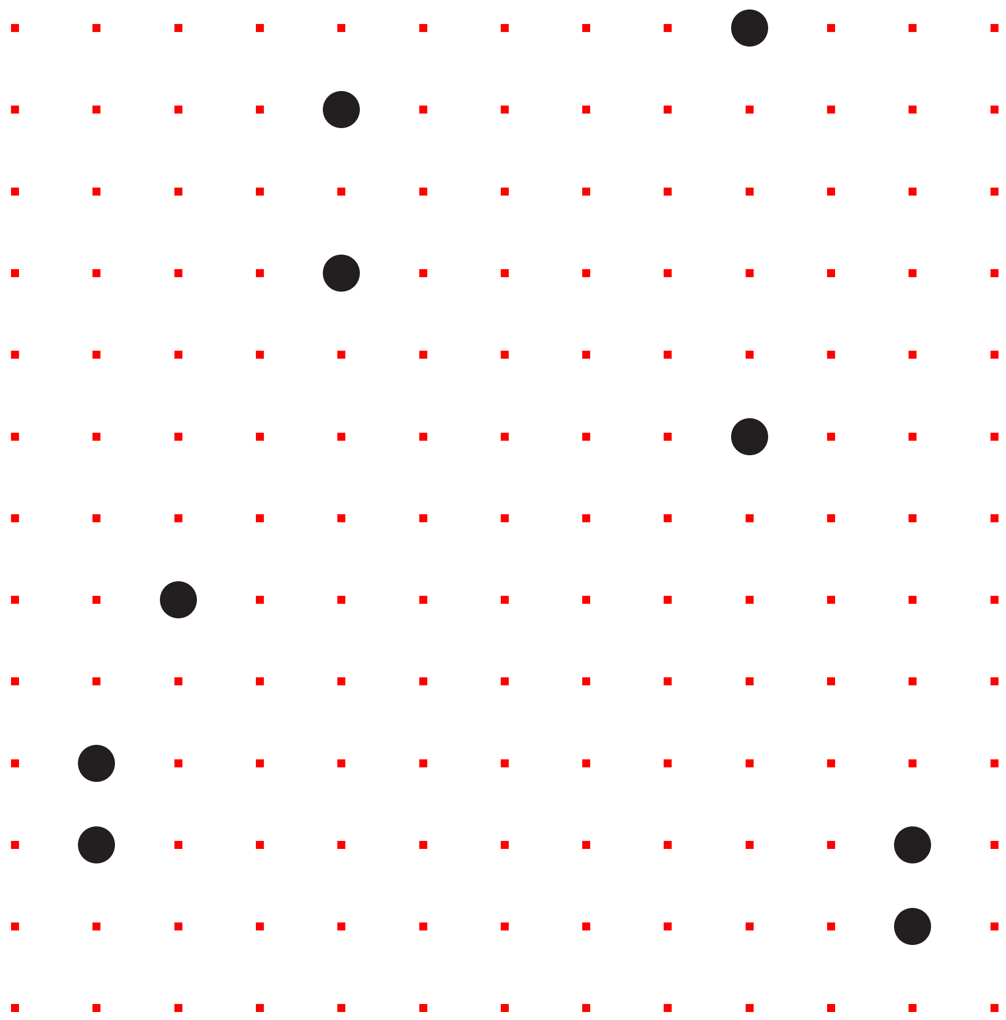
$(x, y) + (x, 5x - y) = \infty$.

An elliptic curve over $\mathbf{Z}/13$

Consider the prime field
$\mathbf{Z}/13 = \{0, 1, 2, \ldots, 12$
with $-, +, \cdot$ defined mod 13.

The "set of points on the elliptic curve $y^2 - 5xy = x^3 - 7$
over $\mathbf{Z}/13$" is
$\{(x, y) \in \mathbf{Z}/13 \times \mathbf{Z}/13 :$
$y^2 - 5xy = x^3 - 7 \ \cup \{\infty$.

Graph of this set of points:



As before, don't forget ∞.

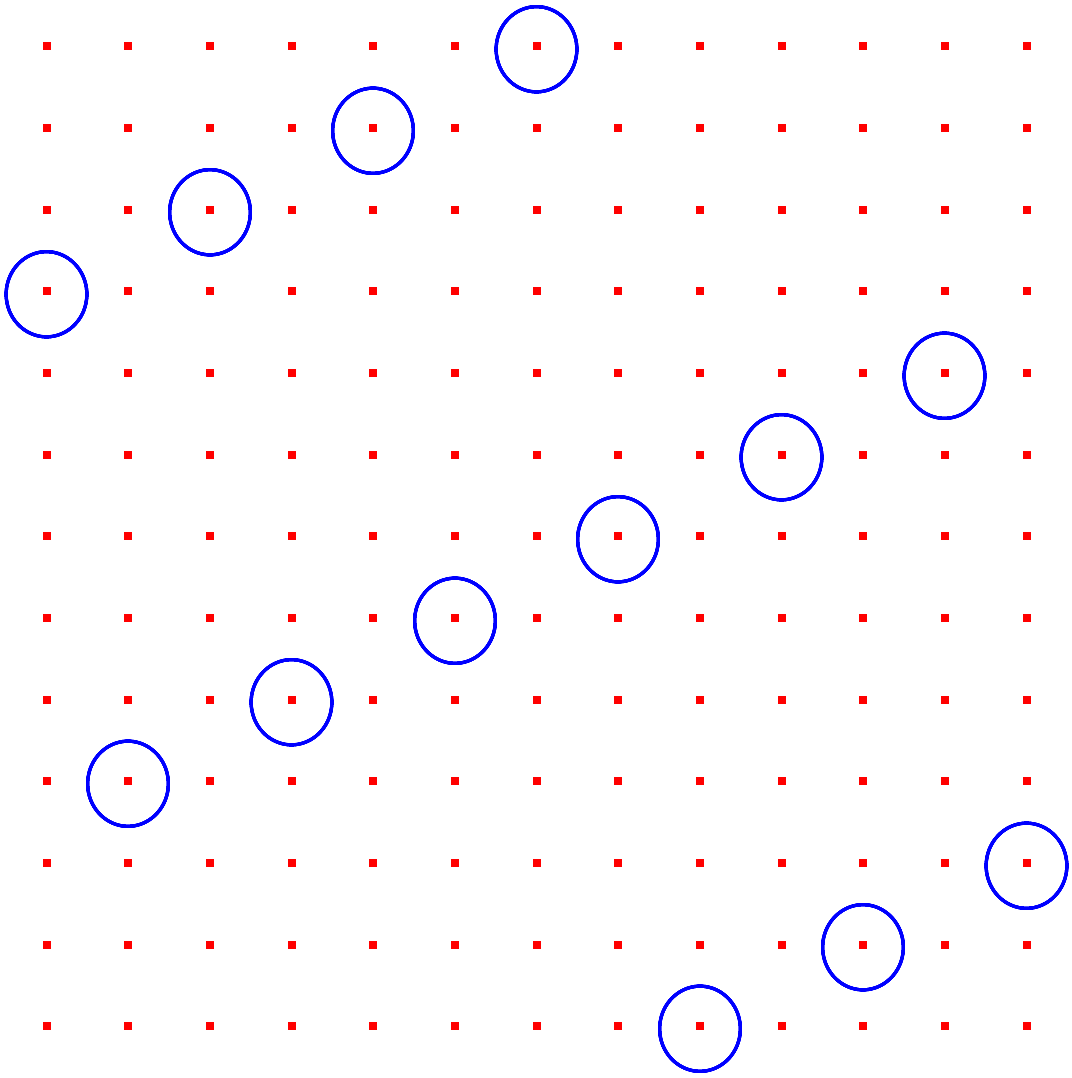
The set of curve points
is a commutative group with
standard definition of $0, -, +$.

Can visualize $0, -, +$ as before.
Replace lines over $\mathbf{R}$
by lines over $\mathbf{Z}/13$.

Warning: tangent is defined by
derivatives; hard to visualize.

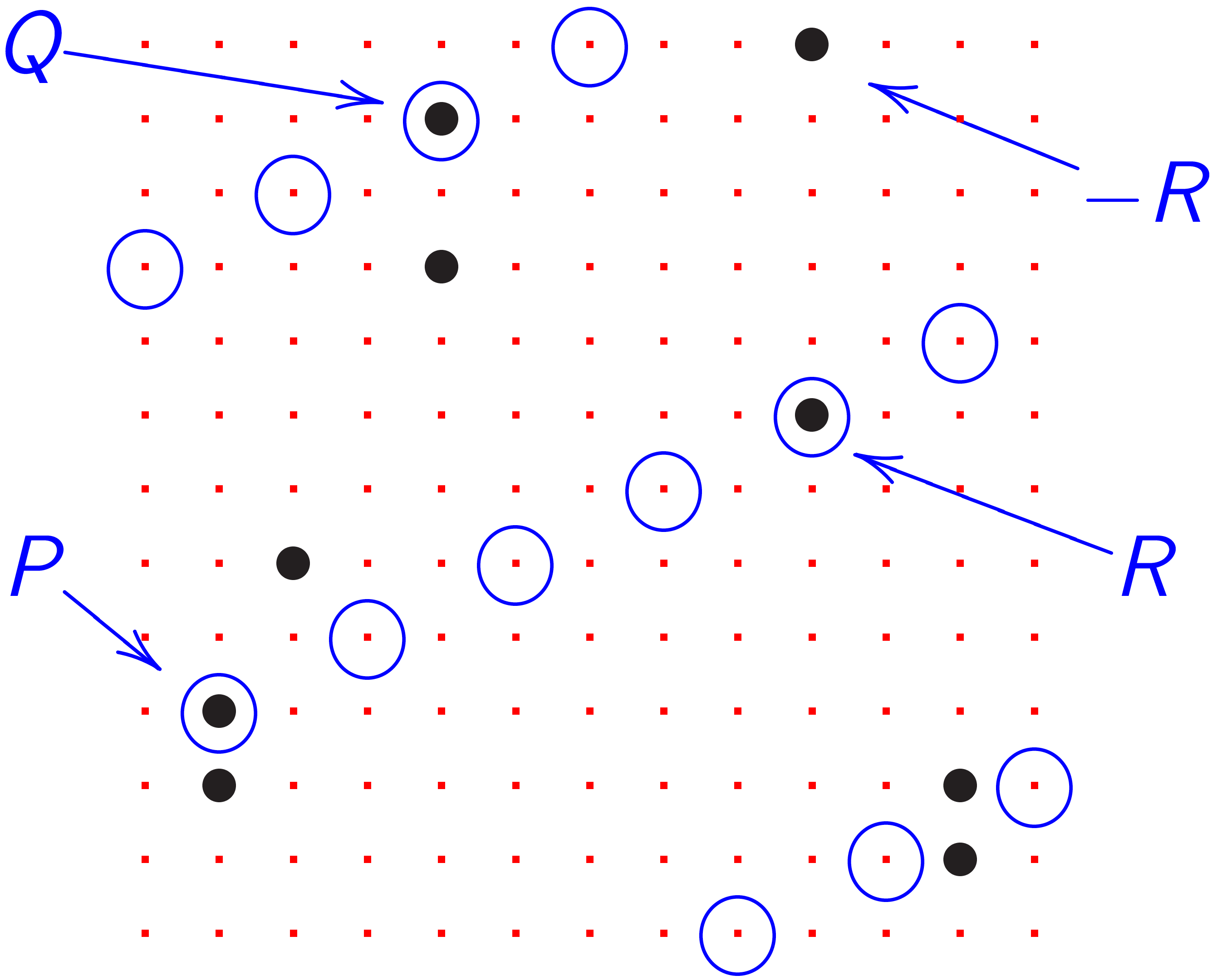
Can define $0, -, +$
using same formulas as before.

Example of line over $\mathbf{Z}/13$:



Formula for this line: $y = 7x + 9$.

$P + Q = -R$:



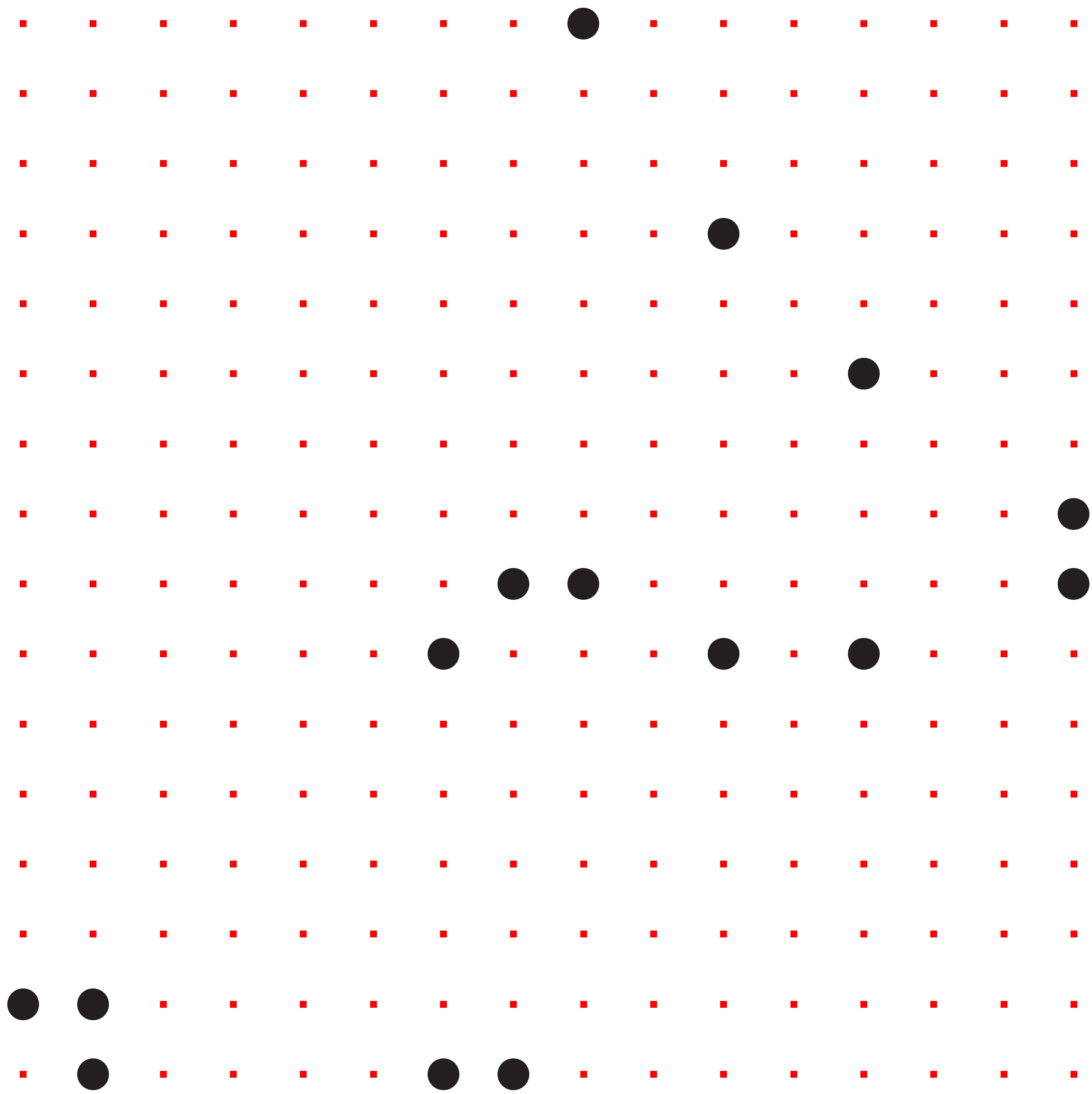
An elliptic curve over $\mathbf{F}_{16}$

Consider the non-prime field

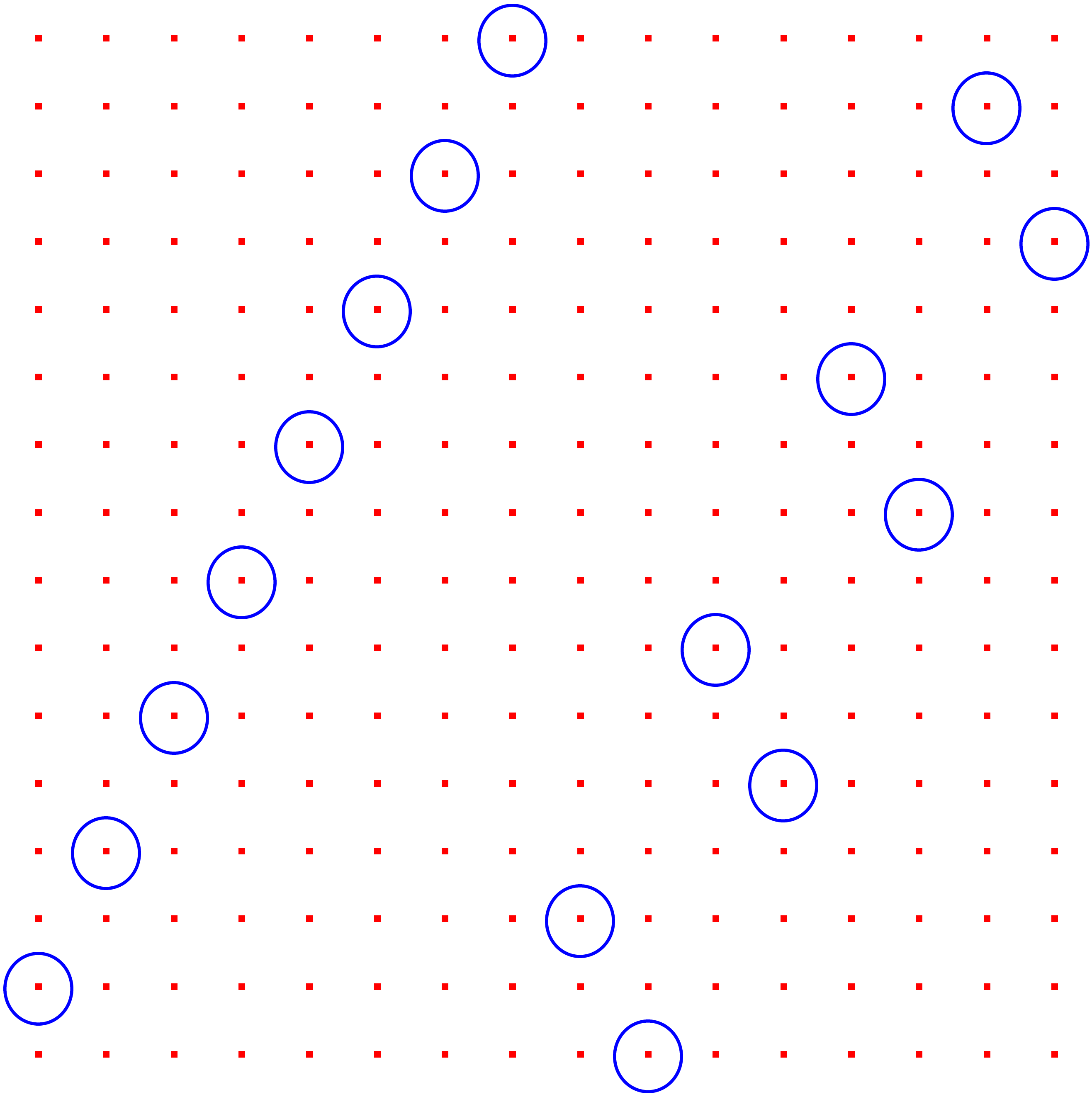
$$(\mathbf{Z}/2)[t]/(t^4 - t - 1) = \{$$
$$0t^3 + 0t^2 + 0t^1 + 0t^0,$$
$$0t^3 + 0t^2 + 0t^1 + 1t^0,$$
$$0t^3 + 0t^2 + 1t^1 + 0t^0,$$
$$0t^3 + 0t^2 + 1t^1 + 1t^0,$$
$$0t^3 + 1t^2 + 0t^1 + 0t^0,$$
$$\vdots$$
$$1t^3 + 1t^2 + 1t^1 + 1t^0$$

of size $2^4 = 16$.

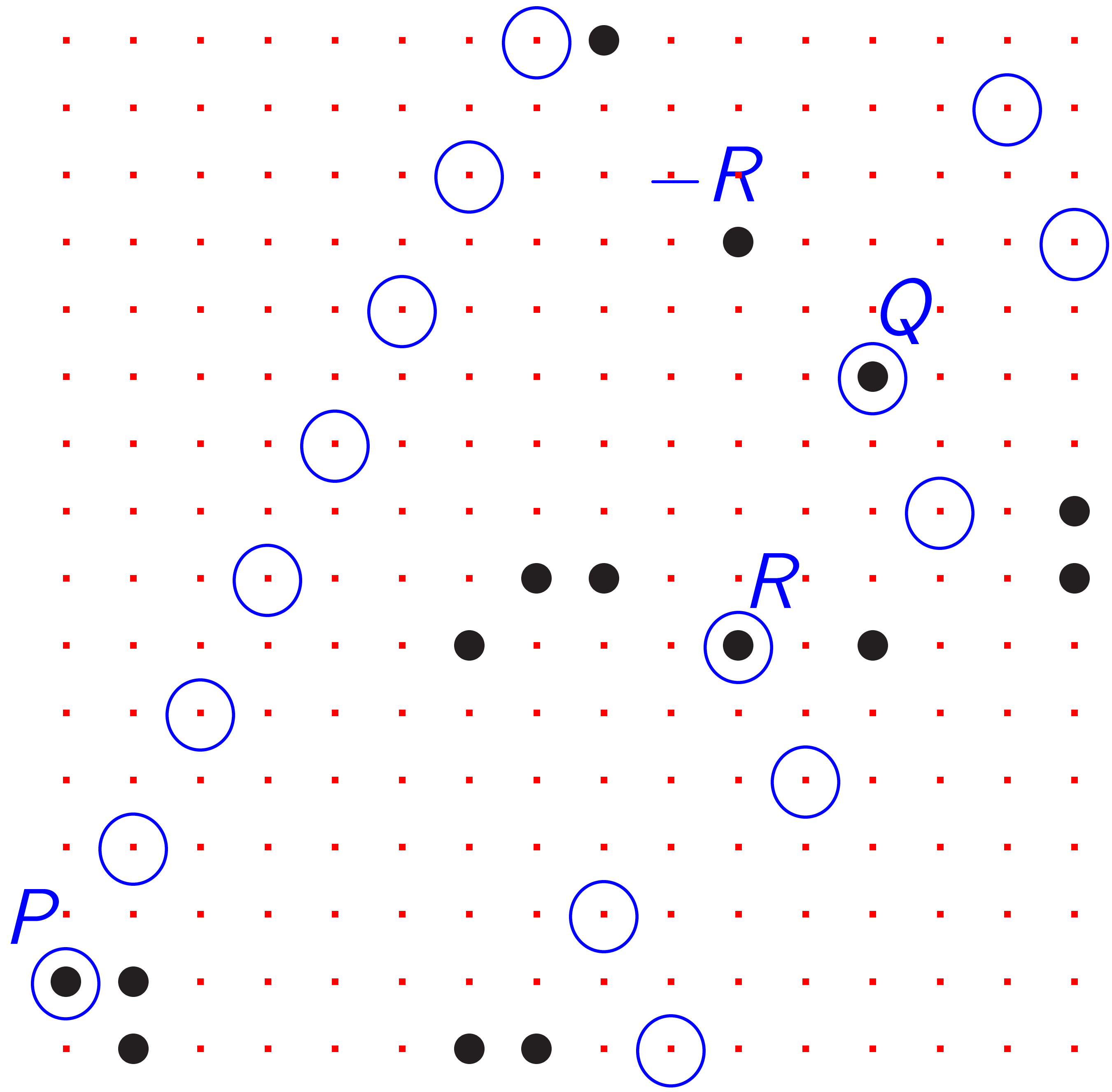
Graph of the "set of points on the elliptic curve $y^2 - 5xy = x^3 - 7$ over $(\mathbf{Z}/2)[t]/(t^4 - t - 1)$":



Line $y = tx + 1$:



$P + Q = -R$:



More elliptic curves

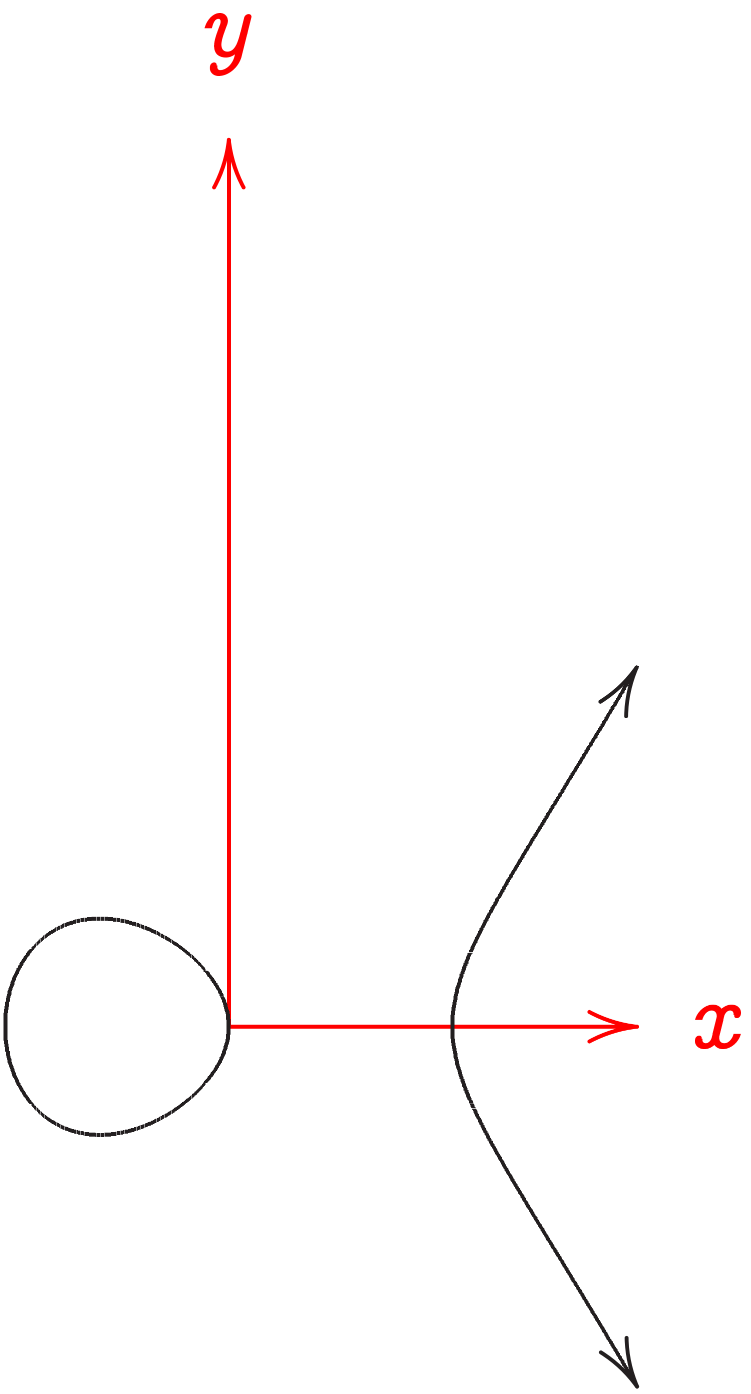
Can use any field k.

Can use any nonsingular curve
$y^2 + a_1xy + a_3y =$
$x^3 + a_2x^2 + a_4x + a_6$.

"Nonsingular": no $(x, y) \in k \times k$ simultaneously satisfies $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ and $2y + a_1x + a_3 = 0$ and $a_1y = 3x^2 + 2a_2x + a_4$.

Easy to check nonsingularity. Almost all curves are nonsingular when k is large.

e.g. $y^2 = x^3 - 30x$:



$\{(x, y) \in k \times k :$
$y^2 + a_1xy + a_3y =$
$x^3 + a_2x^2 + a_4x + a_6\} \cup \{\infty\}$
is a commutative group with standard definition of $0, -, +$.
Points on line add to 0 with appropriate multiplicity.

Group is usually called "$E(k)$" where E is "the elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$."

Fairly easy to write down explicit formulas for $0, -, +$ as before.

If $\#k$ is finite
then $\#E(k)$ is finite.

Each x produces 0, 1, or 2
choices of y with $(x, y) \in E(k)$.
So $1 \le \#E(k) \le 2\#k + 1$;
i.e., $|\#E(k) - \#k - 1| \le \#k$.

Hasse's theorem:
$|\#E(k) - \#k - 1| \le 2\sqrt{\#k}$.

For example, if $k = \mathbf{Z}/1000003$,
then $\#E(k) \in [998004, 1002004]$.

Using explicit formulas can quickly compute nth multiples in $E(k)$ given $n \in \{0, 1, 2, \ldots, 2^{256} - 1\}$ and given E, k with $\#k \approx 2^{256}$.

(How quickly? See Peter Birkner's talk.)

"Elliptic-curve discrete-log problem" (ECDLP): given points P and nP, find n.

Can find curves where ECDLP seems extremely difficult: $\approx 2^{128}$ operations.

See “Handbook of elliptic and hyperelliptic curve cryptography” for much more information.

Two examples of elliptic curves useful for cryptography:

“NIST P-256”: $E(\mathbf{Z}/p)$ where p is the prime $2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$ and E is the elliptic curve $y^2 = x^3 - 3x +$ (a particular constant).

“Curve25519”: $E(\mathbf{Z}/p)$ where p is the prime $2^{255} - 19$ and E is the elliptic curve $y^2 = x^3 + 486662x^2 + x$.