What do quantum computers do?

Daniel J. Bernstein

"Quantum algorithm" means an algorithm that a quantum computer can run.

i.e. a sequence of instructions, where each instruction is in a quantum computer's supported instruction set.

How do we know which instructions a quantum computer will support?

Quantum computer type 1 (QC1):
contains many "qubits";
can efficiently perform
"NOT gate", "Hadamard gate",
"controlled NOT gate", "T gate".

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to compute “Toffoli gate”;
... “Simon’s algorithm”;
... “Shor’s algorithm”; etc.

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to compute “Toffoli gate”;
. . . “Simon’s algorithm”;
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General belief: Traditional CPU isn’t QC1; e.g. can’t factor quickly.

Quantum computer type 2 (QC2): stores a simulated universe; efficiently simulates the laws of quantum physics with as much accuracy as desired.

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General belief: any QC1 is a QC2. Partial proof: see, e.g., 2011 Jordan–Lee–Preskill "Quantum algorithms for quantum field theories".

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General belief: any QC3 is a QC1.
Argument for belief:
look, we're building a QC1.

A note on D-Wave

Apparent scientific consensus:
Current "quantum computers" from D-Wave are useless—can be more cost-effectively simulated by traditional CPUs.

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Is D-Wave a bad investment?

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a list of 3 elements of $\{0, 1\}$.
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$0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0,$
$0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1,$
$1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1,$
$0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0,$
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Data stored in 4 qubits: a list of 16 numbers, not all zero. e.g.:
$(3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3)$.

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

Data stored in 1000 qubits: a list of 2^{1000} numbers, not all zero.

Measuring a quantum computer

Can simply look at a bit.

Cannot simply look at the list of numbers stored in n qubits.

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State is then all zeros except 1 at position q.

e.g.: Say 3 qubits have state $(1, 1, 1, 1, 1, 1, 1, 1)$.

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Measurement produces
$000 = 0$ with probability $1/8$;
$001 = 1$ with probability $1/8$;
$010 = 2$ with probability $1/8$;
$011 = 3$ with probability $1/8$;
$100 = 4$ with probability $1/8$;
$101 = 5$ with probability $1/8$;
$110 = 6$ with probability $1/8$;
$111 = 7$ with probability $1/8$.

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"Quantum RNG."

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"Quantum RNG."

Warning: Quantum RNGs sold today are measurably biased.

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Measurement produces
$000 = 0$ with probability $9/173$;
$001 = 1$ with probability $1/173$;
$010 = 2$ with probability $16/173$;
$011 = 3$ with probability $1/173$;
$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

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$000 = 0$ with probability $9/173$;
$001 = 1$ with probability $1/173$;
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$011 = 3$ with probability $1/173$;
$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

5 is most likely outcome.

e.g.: Say 3 qubits have state
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$000 = 0$ with probability 0;
$001 = 1$ with probability 0;
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Measurement produces
$000 = 0$ with probability 0;
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$010 = 2$ with probability 0;
$011 = 3$ with probability 0;
$100 = 4$ with probability 0;
$101 = 5$ with probability 1;
$110 = 6$ with probability 0;
$111 = 7$ with probability 0.

5 is guaranteed outcome.

NOT gates

NOT_0 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(1, 3, 1, 4, 9, 5, 6, 2)$.

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NOT_0 gate on 4 qubits:
$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
$(1,3,1,4,9,5,6,2,3,5,8,5,7,9,3,9)$.

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NOT_1 gate on 3 qubits:

$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT gates

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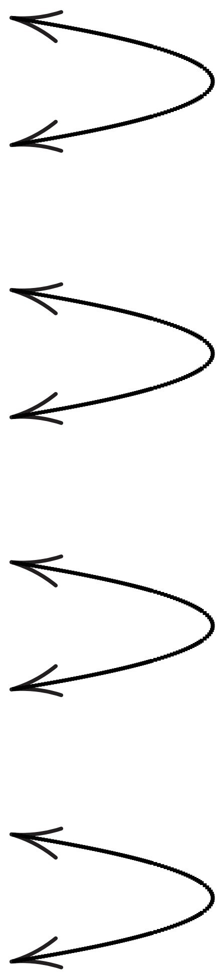
NOT_1 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT_2 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(5, 9, 2, 6, 3, 1, 4, 1)$.

state	measurement
(1, 0, 0, 0, 0, 0, 0, 0)	000
(0, 1, 0, 0, 0, 0, 0, 0)	001
(0, 0, 1, 0, 0, 0, 0, 0)	010
(0, 0, 0, 1, 0, 0, 0, 0)	011
(0, 0, 0, 0, 1, 0, 0, 0)	100
(0, 0, 0, 0, 0, 1, 0, 0)	101
(0, 0, 0, 0, 0, 0, 1, 0)	110
(0, 0, 0, 0, 0, 0, 0, 1)	111



Operation on quantum state:
NOT_0, swapping pairs.

Operation after measurement:
flipping bit 0 of result.

Flip: output is not input.

Controlled-NOT (CNOT) gates

e.g. C_1NOT_0:

$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$

$(3, 1, 1, 4, 5, 9, 6, 2)$.

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flipping bit 0 *if* bit 1 is set; i.e.,
$(q_2, q_1, q_0) \mapsto (q_2, q_1, q_0 \oplus q_1)$.

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e.g. C_2NOT_0:
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$(3,1,4,1,9,5,6,2)$.

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e.g. C_2NOT_0:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
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e.g. C_0NOT_2:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(3, 9, 4, 6, 5, 1, 2, 1)$.

Toffoli gates

Also known as CCNOT gates: controlled-controlled-NOT gates.

e.g. $C_2C_1NOT_0$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
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Operation after measurement:
$(q_2, q_1, q_0) \mapsto (q_2, q_1, q_0 \oplus q_1 q_2)$.

e.g. $C_0C_1NOT_2$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto (3, 1, 4, 6, 5, 9, 2, 1)$.

More shuffling

Combine NOT, CNOT, Toffoli
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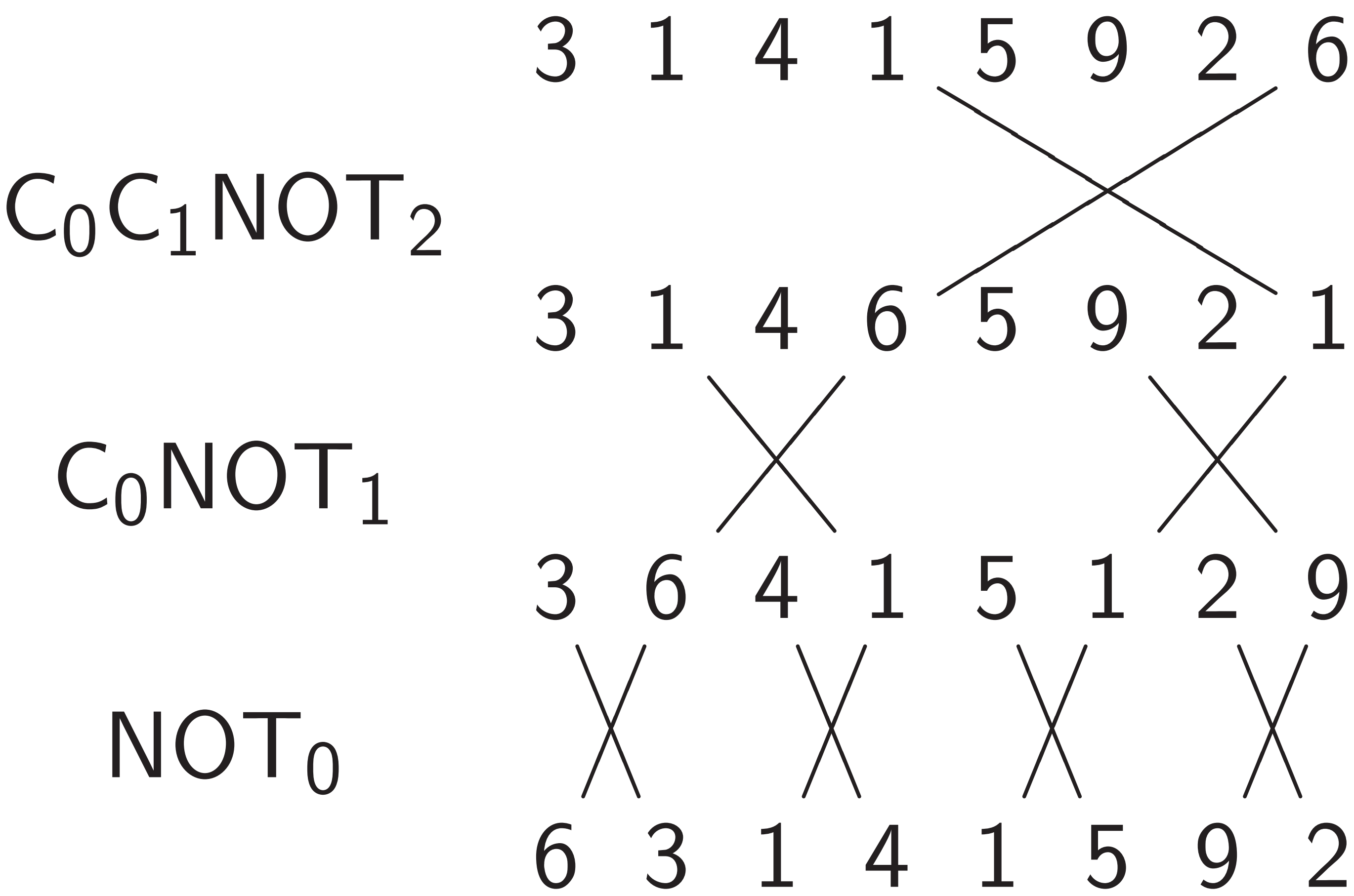
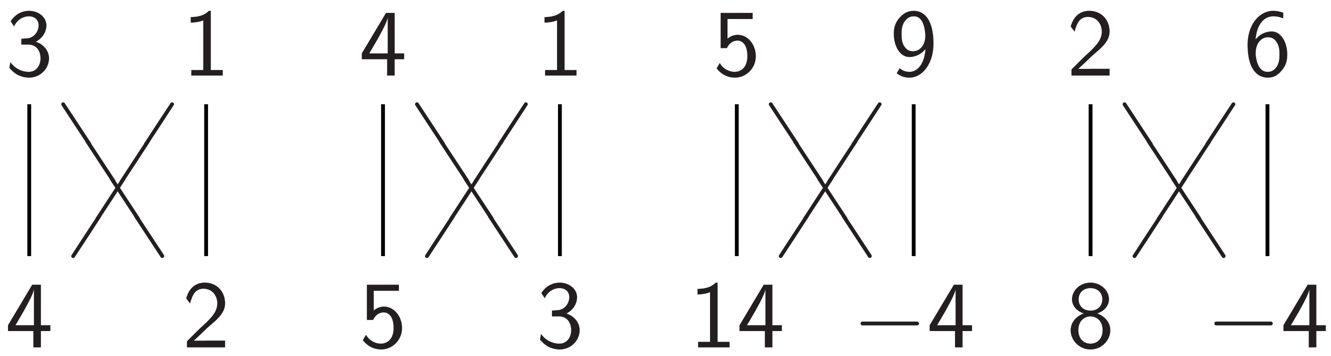
e.g. series of gates to
rotate 8 positions by distance 1:



Hadamard gates

Hadamard$_0$:

$(a, b) \mapsto (a + b, a - b)$.



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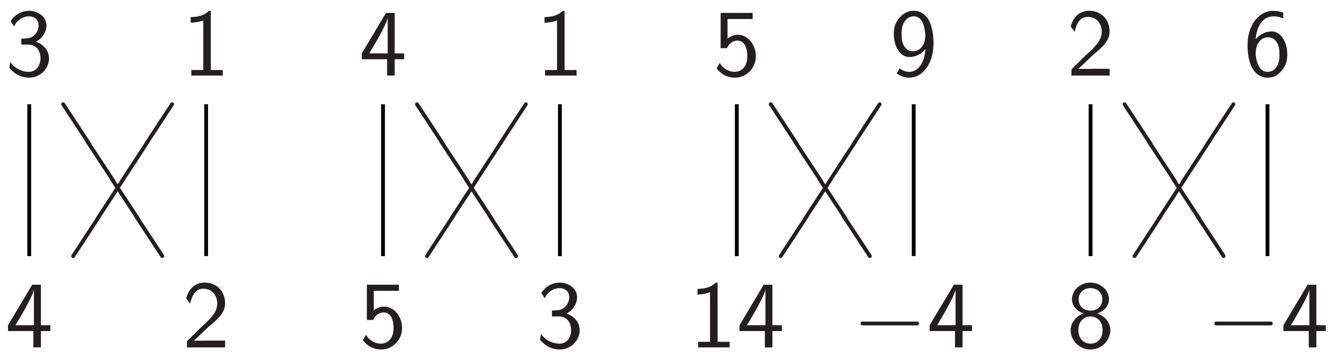
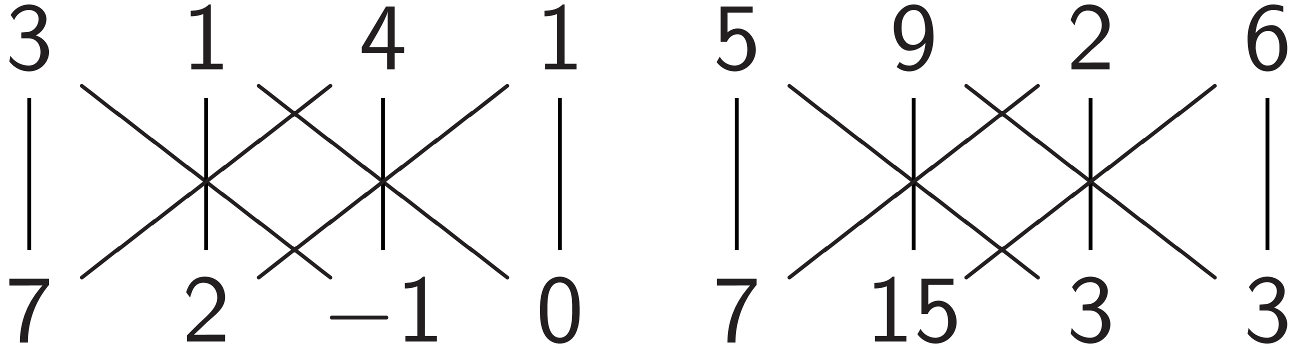


Hadamard$_1$:

$(a, b, c, d) \mapsto$
$(a + c, b + d, a - c, b - d)$.



Some Hadamard applications

Hadamard$_0$, NOT$_0$, Hadamard$_0$:

3	1	4	1	5	9	2	6
4	2	5	3	14	−4	8	−4
2	4	3	5	−4	14	−4	8
6	−2	8	−2	10	−18	4	−12

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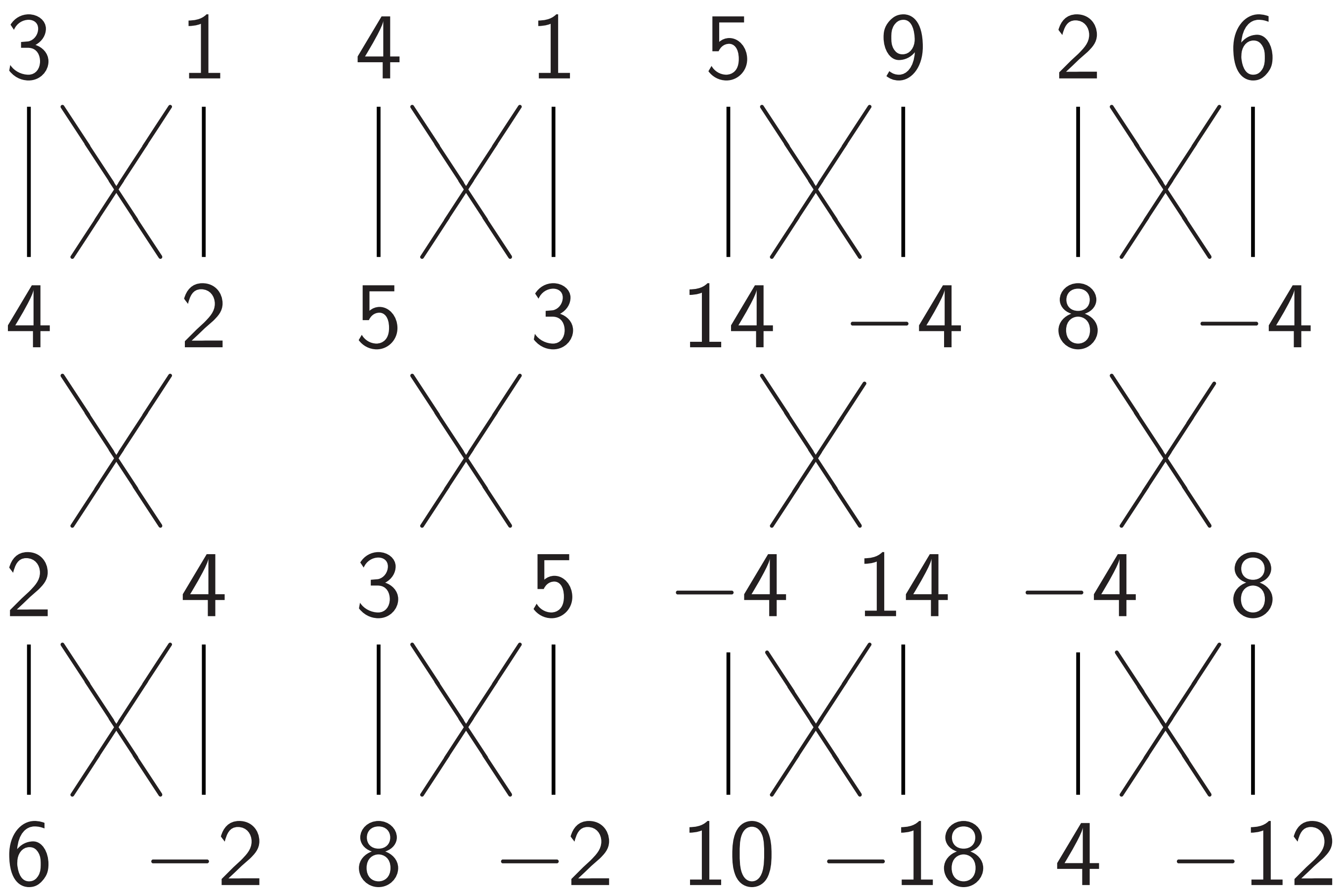

"Multiply each amplitude by 2."

This is not physically observable.

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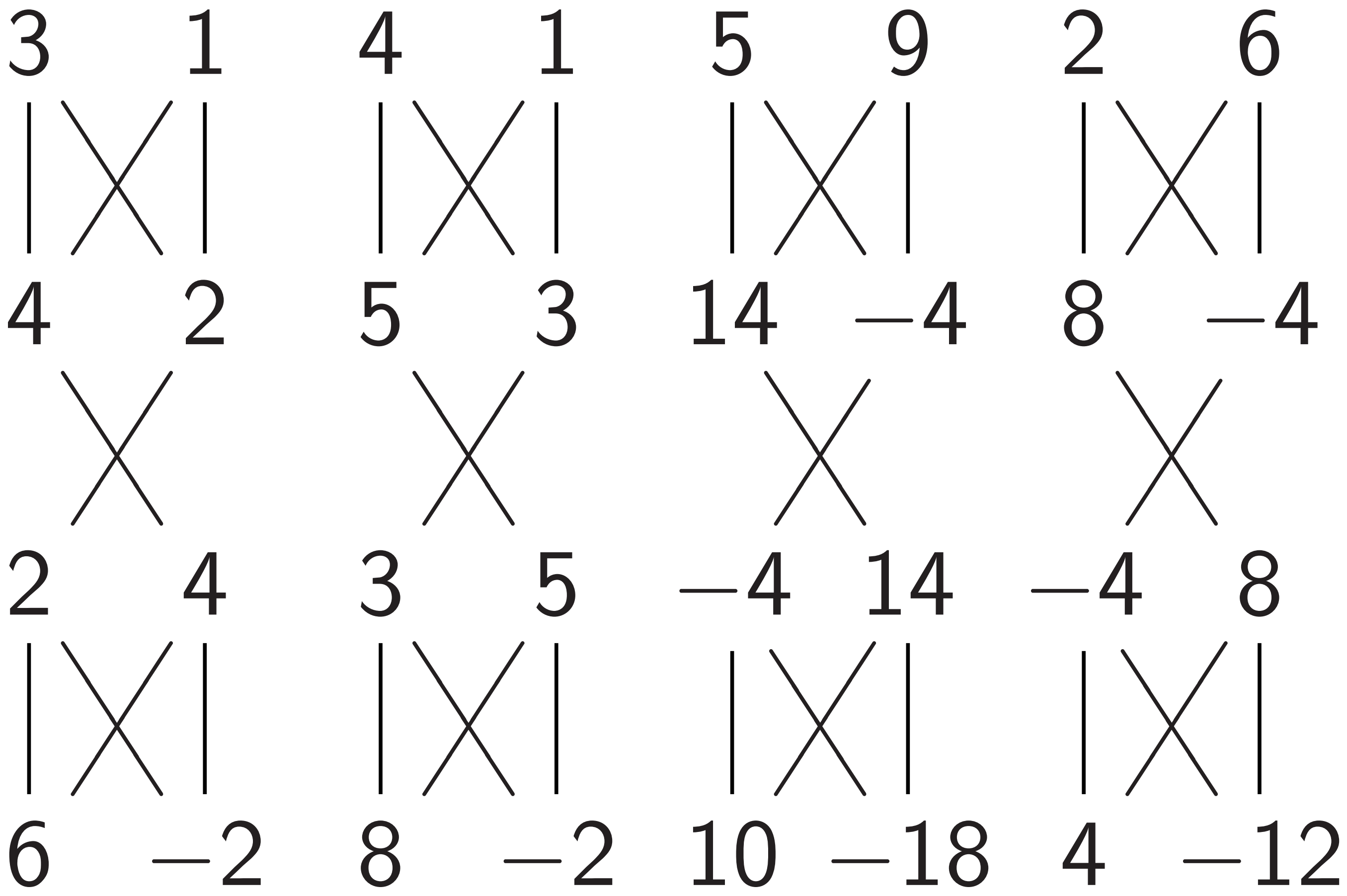


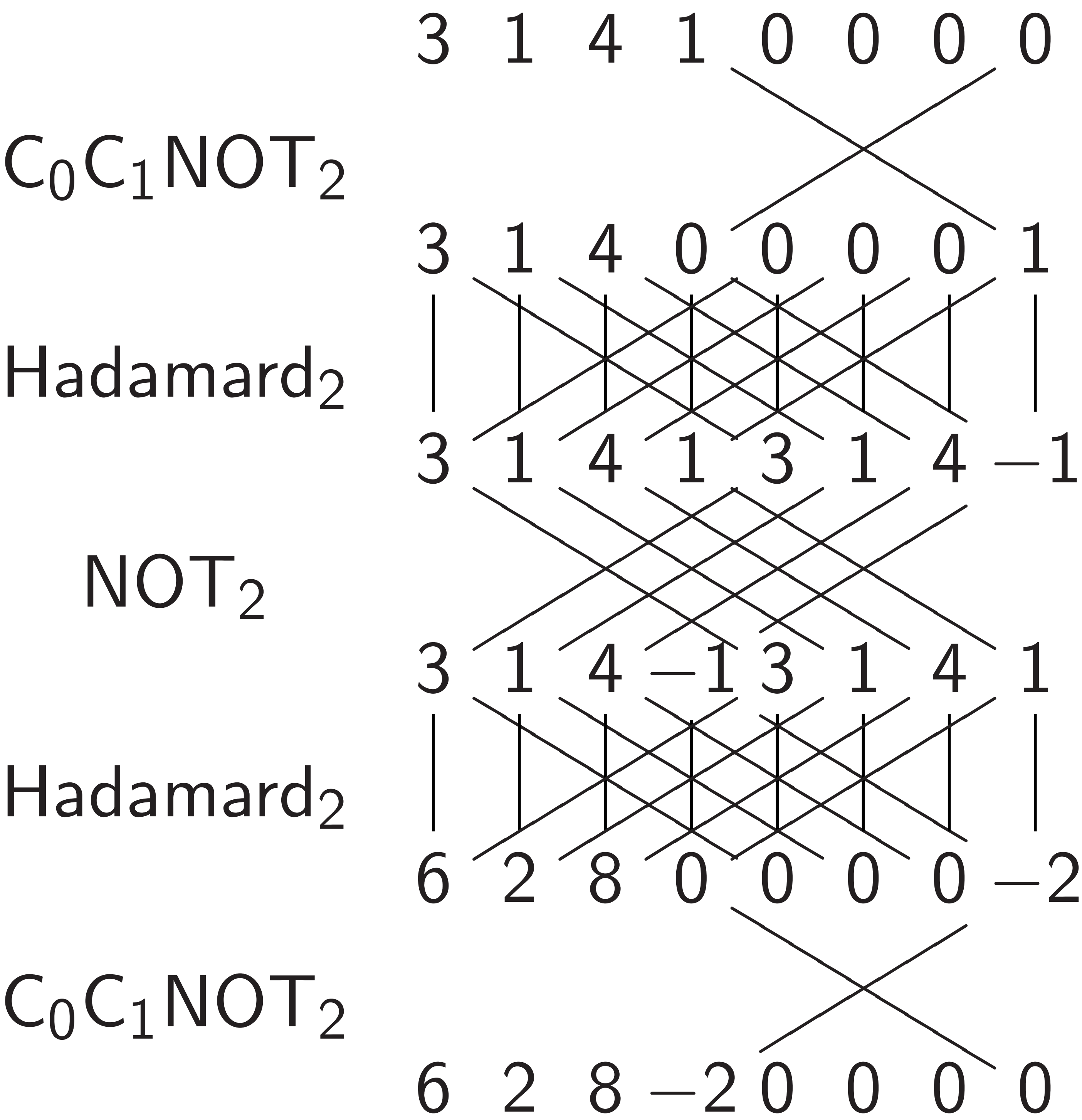
"Multiply each amplitude by 2."
This is not physically observable.

"Negate amplitude if q_0 is set."
No effect on measuring *now*.

Fancier example:

"Negate amplitude if $q_0 q_1$ is set."

Assumes $q_2 = 0$: "ancilla" qubit.



Affects measurements: “Negate amplitude around its average.”

$(3, 1, 4, 1) \mapsto (1.5, 3.5, 0.5, 3.5)$.

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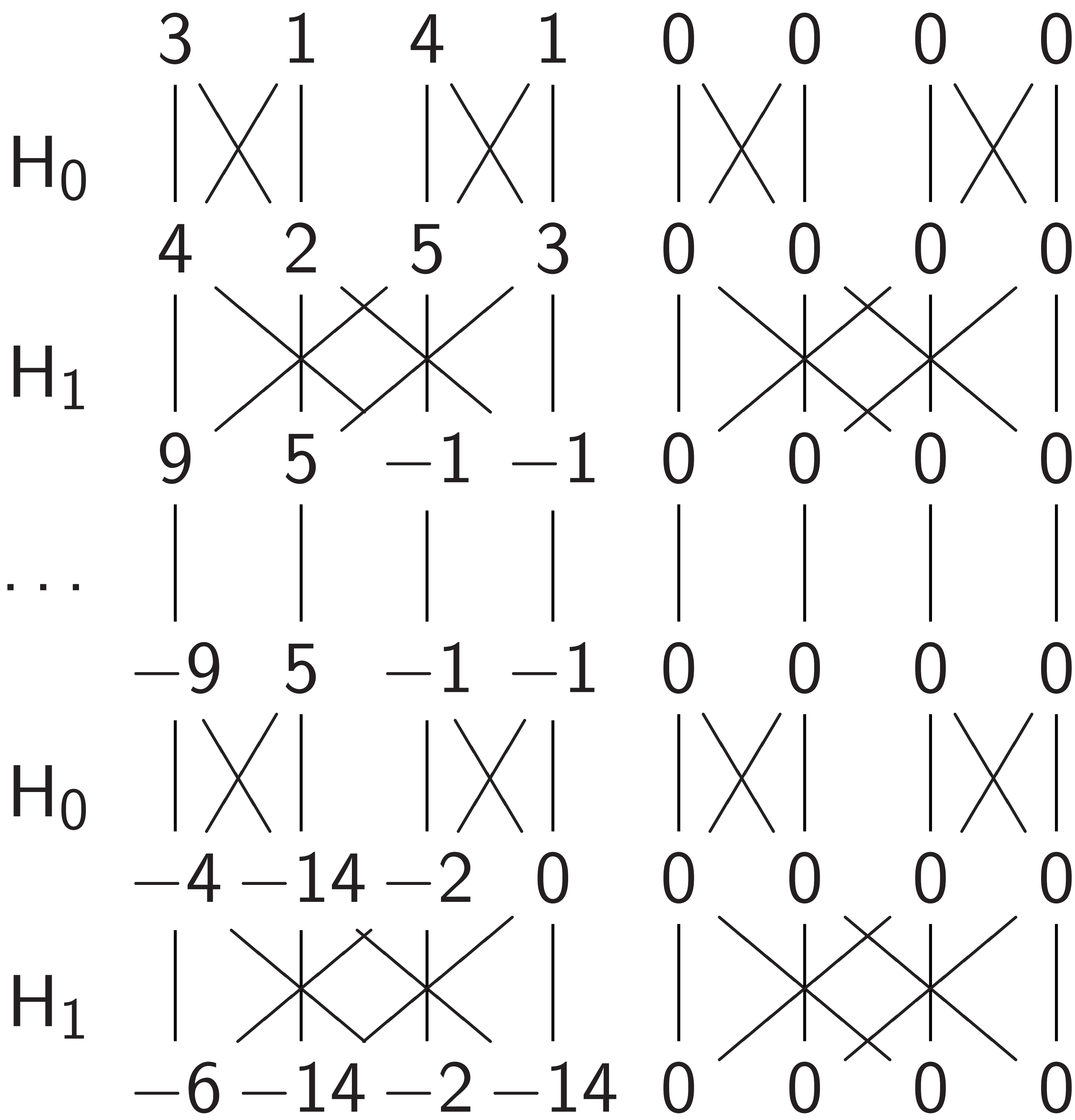

Simon's algorithm

Step 1. Set up pure zero state:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 2. Hadamard$_0$:

1, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 3. Hadamard$_1$:

1, 1, 1, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 4. Hadamard$_2$:

1, 1, 1, 1, 1, 1, 1, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe.

Simon's algorithm

Step 5. C_0NOT_3:

1, 0, 1, 0, 1, 0, 1, 0,
0, 1, 0, 1, 0, 1, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5b. More shuffling:

1, 0, 0, 0, 1, 0, 0, 0,
0, 1, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 1, 0,
0, 0, 0, 1, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5c. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 1, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5d. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5e. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5f. More shuffling:

0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5g. More shuffling:

0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5h. More shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5i. More shuffling:

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 1, 0, 0, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 1, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 1, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Surprise: u and $u \oplus 101$ match.

Simon's algorithm

Step 6. Hadamard$_0$:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, $\overline{1}$, 0, 0, 1, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 1, 0, 0, 1, $\overline{1}$,
1, $\overline{1}$, 0, 0, 1, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
1, 1, 0, 0, 1, $\overline{1}$, 0, 0.

Notation: $\overline{1}$ means -1.

Simon's algorithm

Step 7. Hadamard$_1$:

$0, 0, 0, 0, 0, 0, 0, 0,$
$1, \overline{1}, \overline{1}, 1, 1, 1, \overline{1}, \overline{1},$
$0, 0, 0, 0, 0, 0, 0, 0,$
$1, 1, \overline{1}, \overline{1}, 1, \overline{1}, \overline{1}, 1,$
$1, \overline{1}, 1, \overline{1}, 1, 1, 1, 1,$
$0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0,$
$1, 1, 1, 1, 1, \overline{1}, 1, \overline{1}.$

Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Step 9: Measure. Obtain some information about the surprise: a random vector orthogonal to 101.

Repeat to figure out 101.

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Generalize Step 5 to any function $u \mapsto f(u)$ with $f(u) = f(u \oplus s)$.
"Usually" algorithm figures out s.

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Shor's algorithm replaces $\oplus$ with more general $+$ operation.
Many spectacular applications.

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e.g. Shor finds "random" s with $2^u \bmod N = 2^{u+s} \bmod N$.
Easy to factor N using this.

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e.g. Shor finds "random" s, t with $4^u 9^v \bmod p = 4^{u+s} 9^{v+t} \bmod p$. Easy to compute discrete logs.

Grover's algorithm

Assume: unique $s \in \{0,1\}^n$ has $f(s) = 0$.

Traditional algorithm to find s: compute f for many inputs, hope to find output 0.
Success probability is very low until #inputs approaches 2^n.

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Traditional algorithm to find s: compute f for many inputs, hope to find output 0. Success probability is very low until #inputs approaches 2^n.

Grover's algorithm takes only $2^{n/2}$ reversible computations of f. Typically: reversibility overhead is small enough that this easily beats traditional algorithm.

Start from uniform superposition over all n-bit strings q.

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Step 1: Set $a \leftarrow b$ where
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$b_q = a_q$ otherwise.
This is fast.

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Negate a around its average.
This is also fast.

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Repeat Step 1 + Step 2 about $0.58 \cdot 2^{0.5n}$ times.

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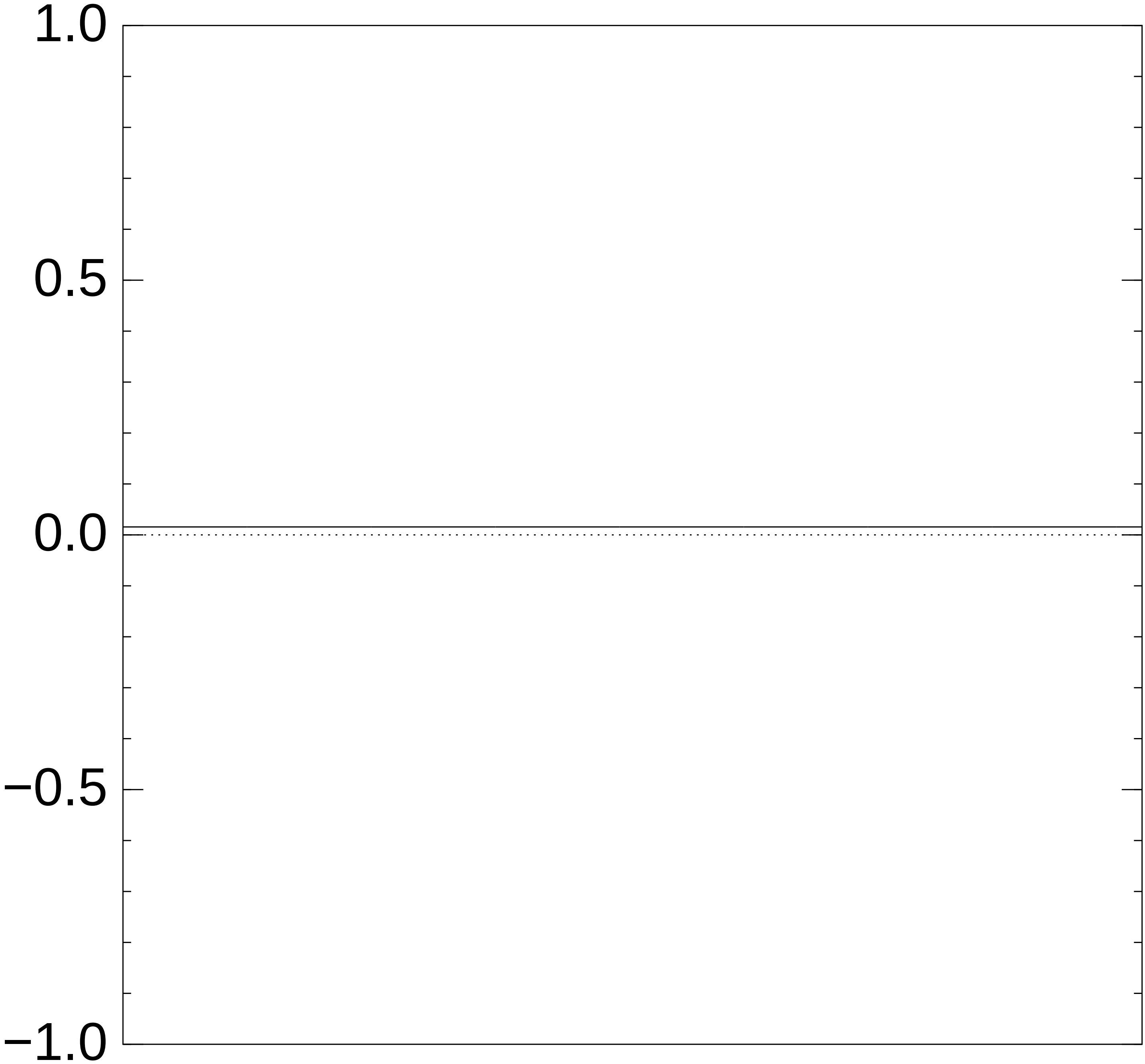
This is fast.

Step 2: "Grover diffusion".
Negate a around its average.
This is also fast.

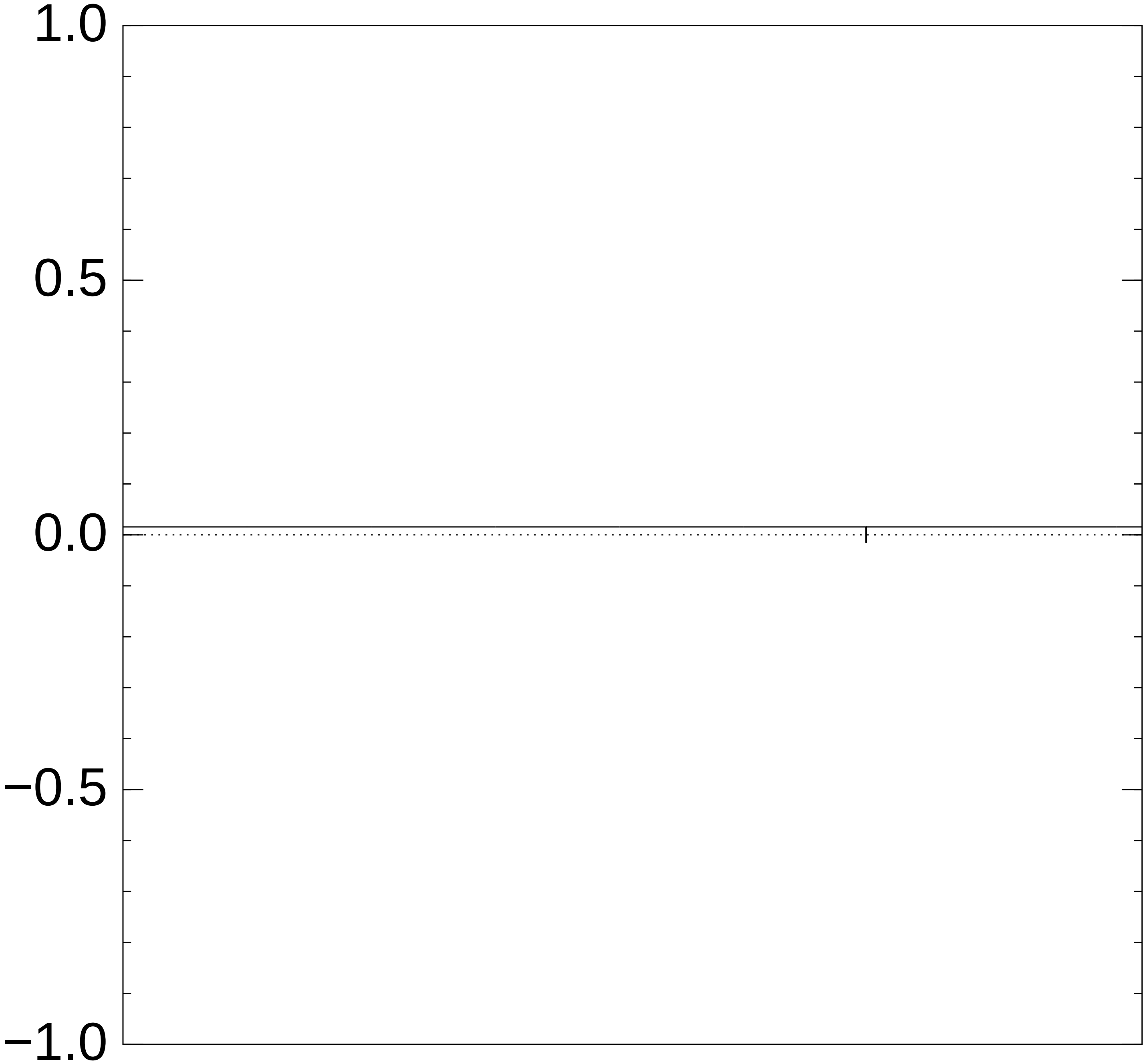
Repeat Step 1 + Step 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds s.

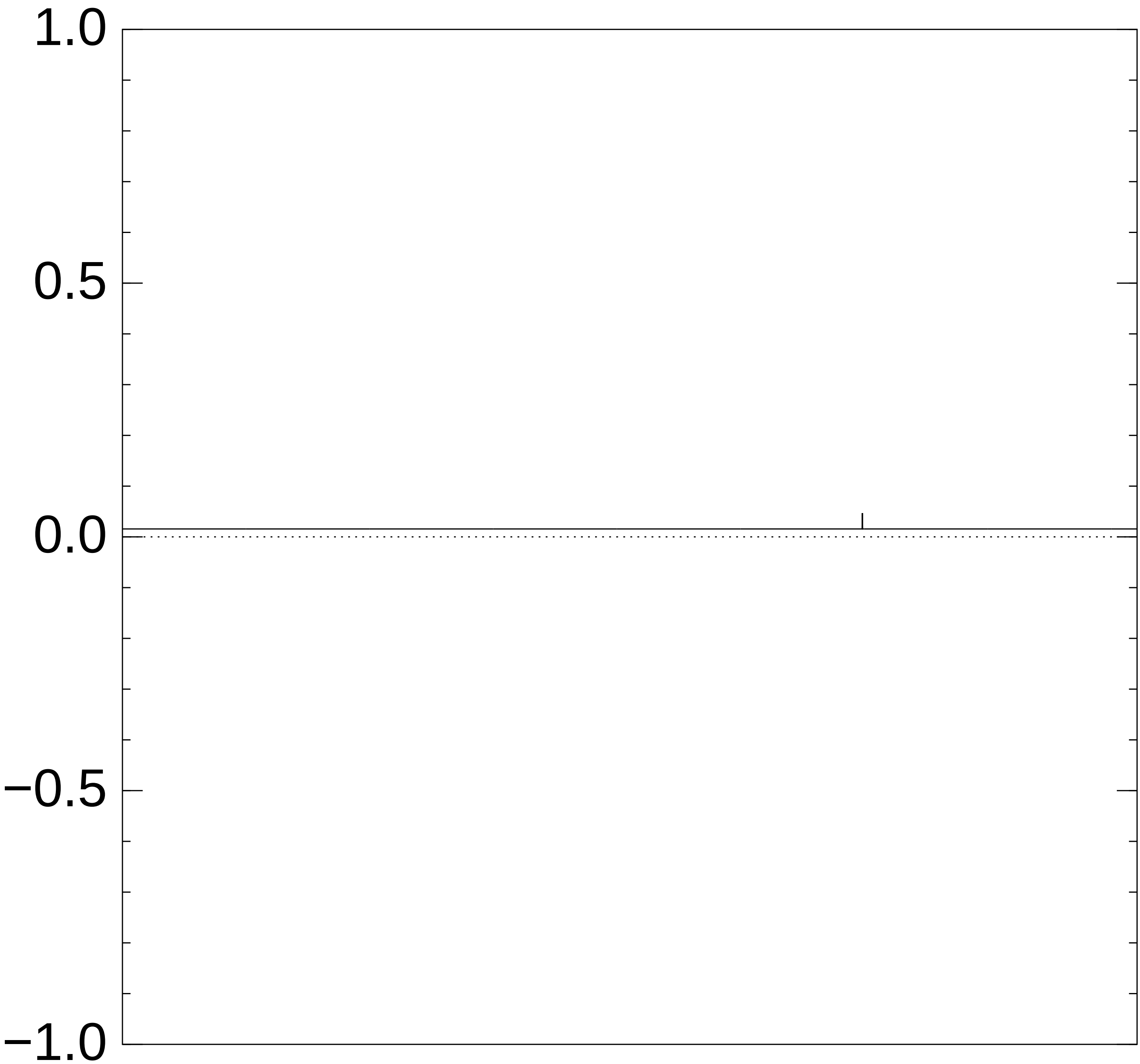
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after 0 steps:



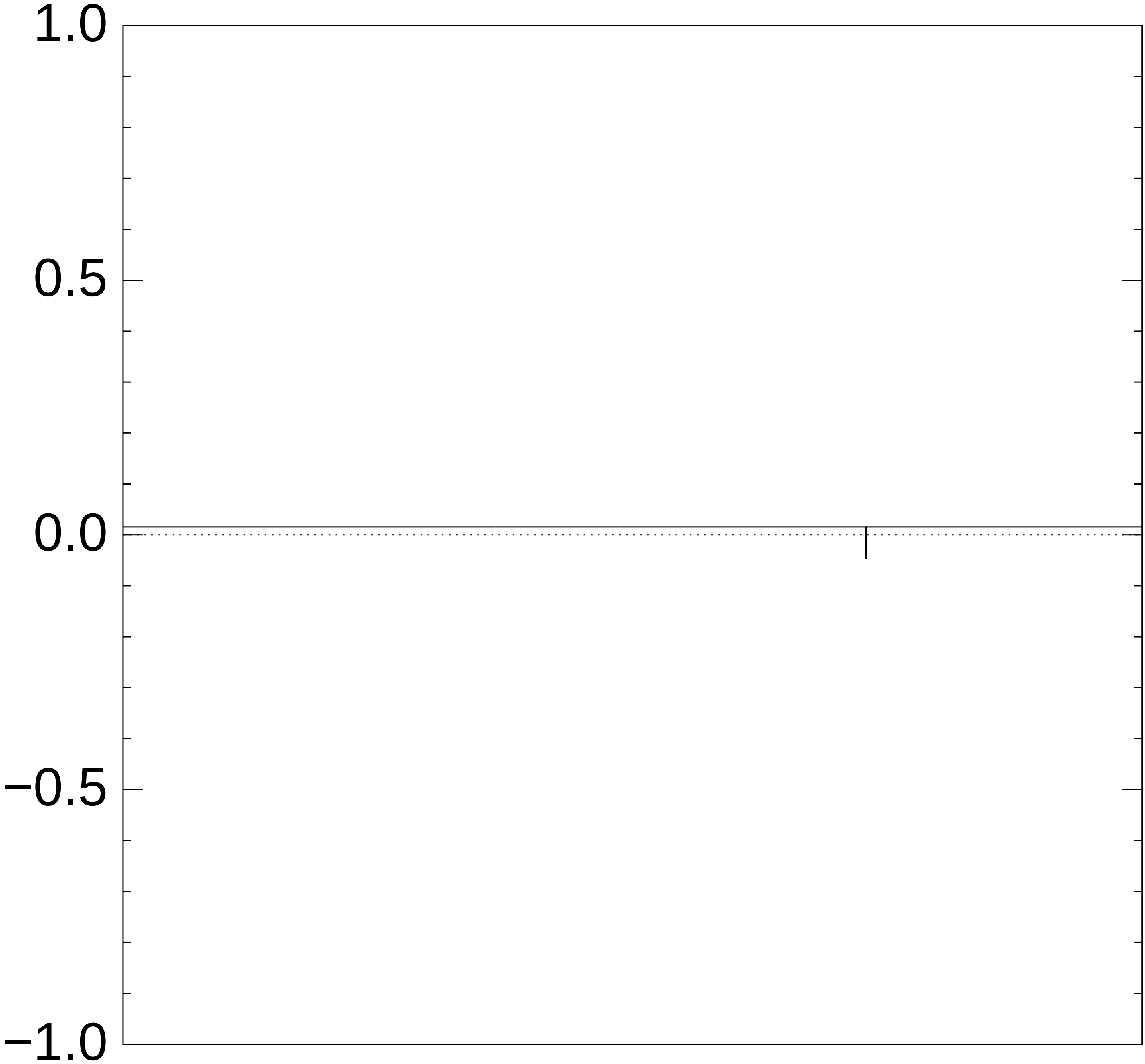
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1:



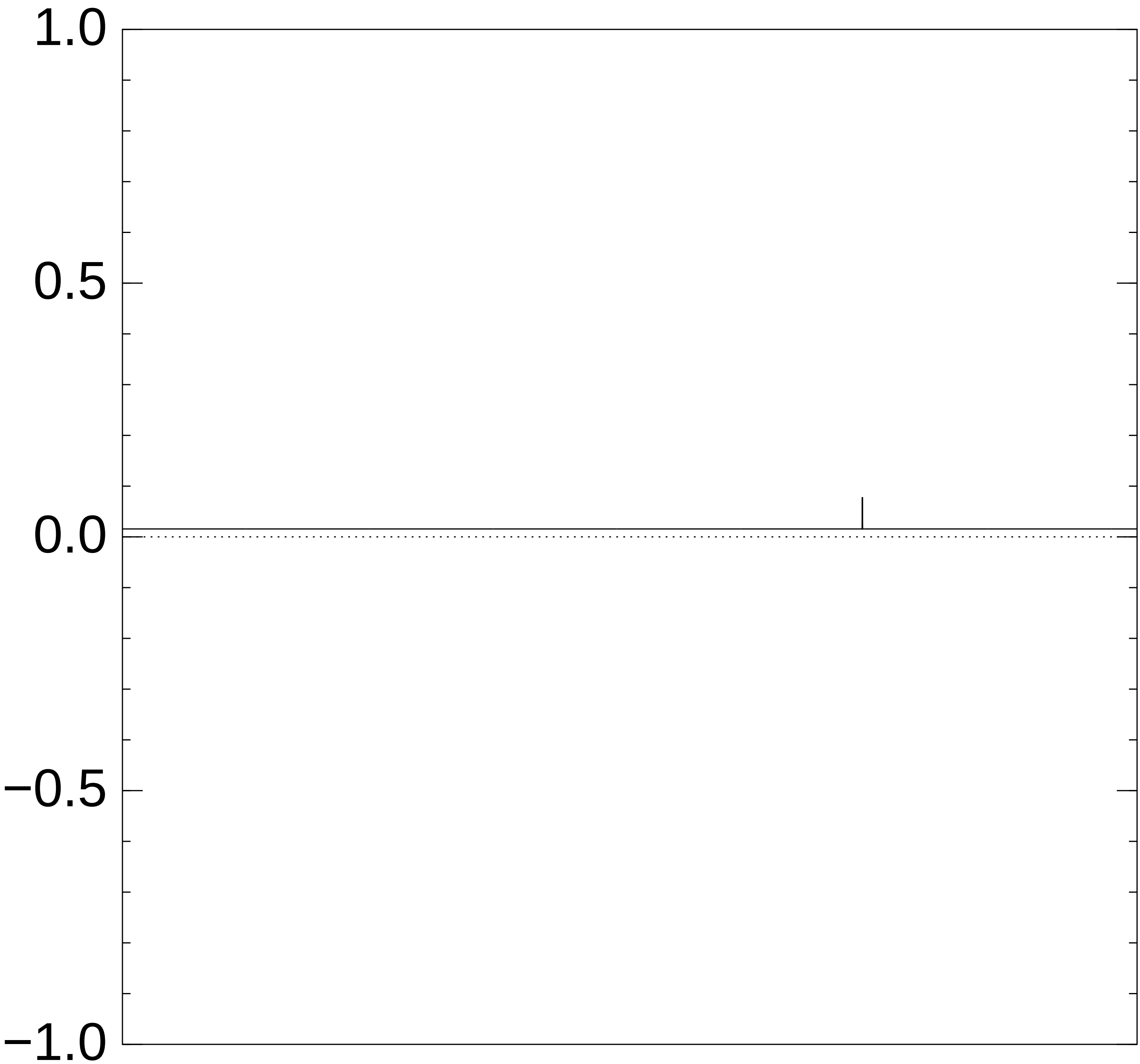
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after Step 1 + Step 2:



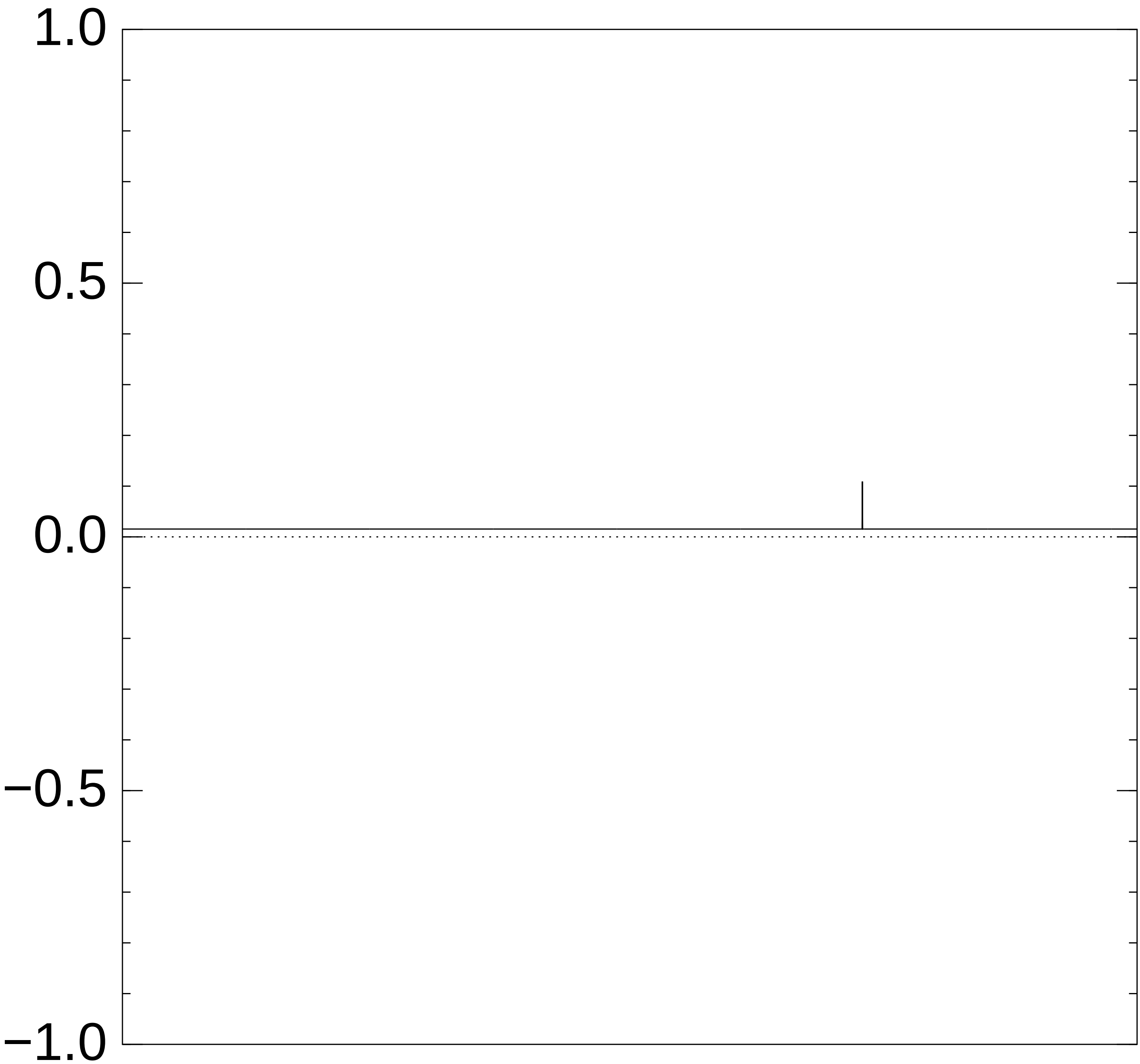
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1 + Step 2 + Step 1:



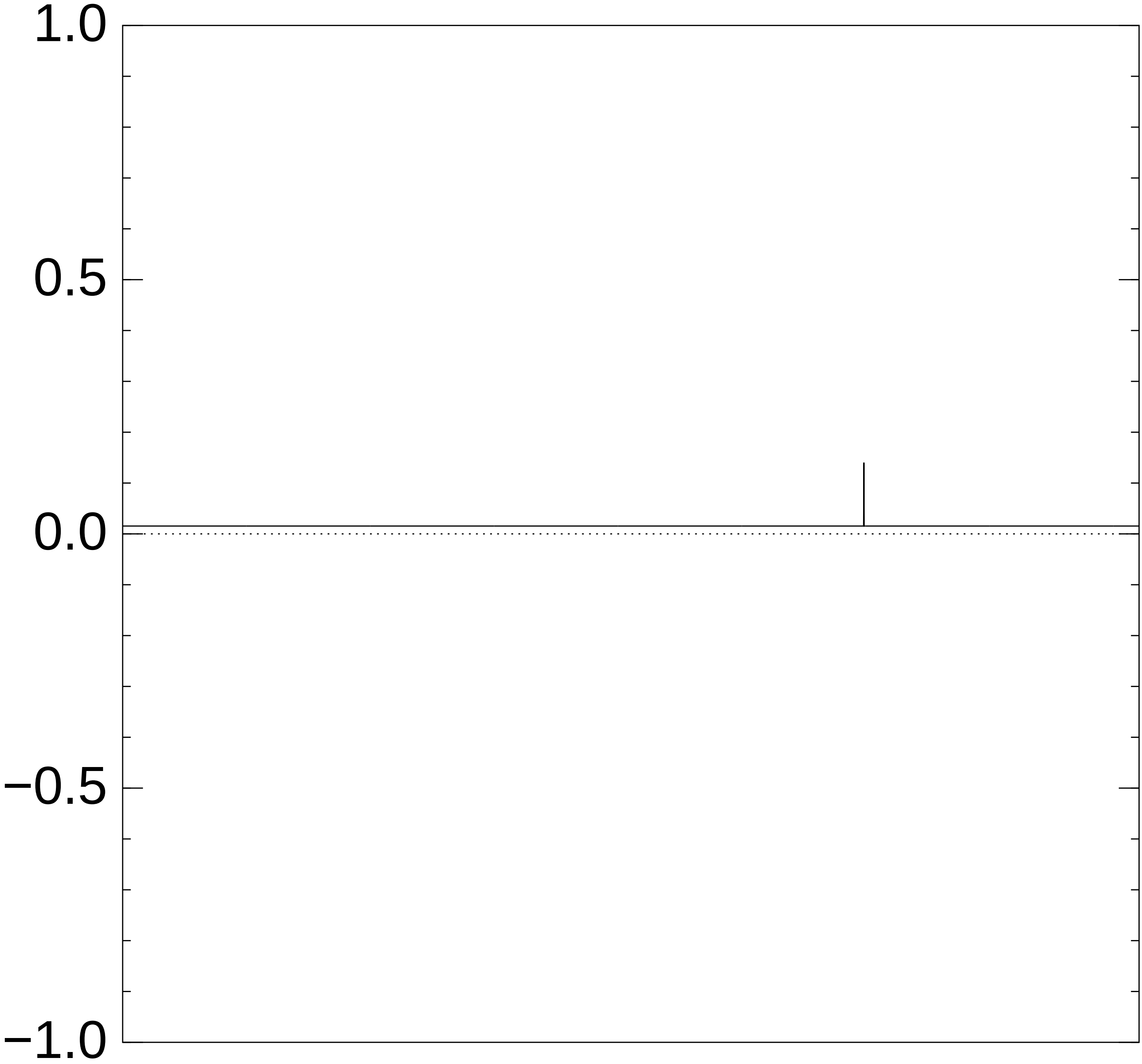
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $2 \times (\text{Step } 1 + \text{Step } 2)$:



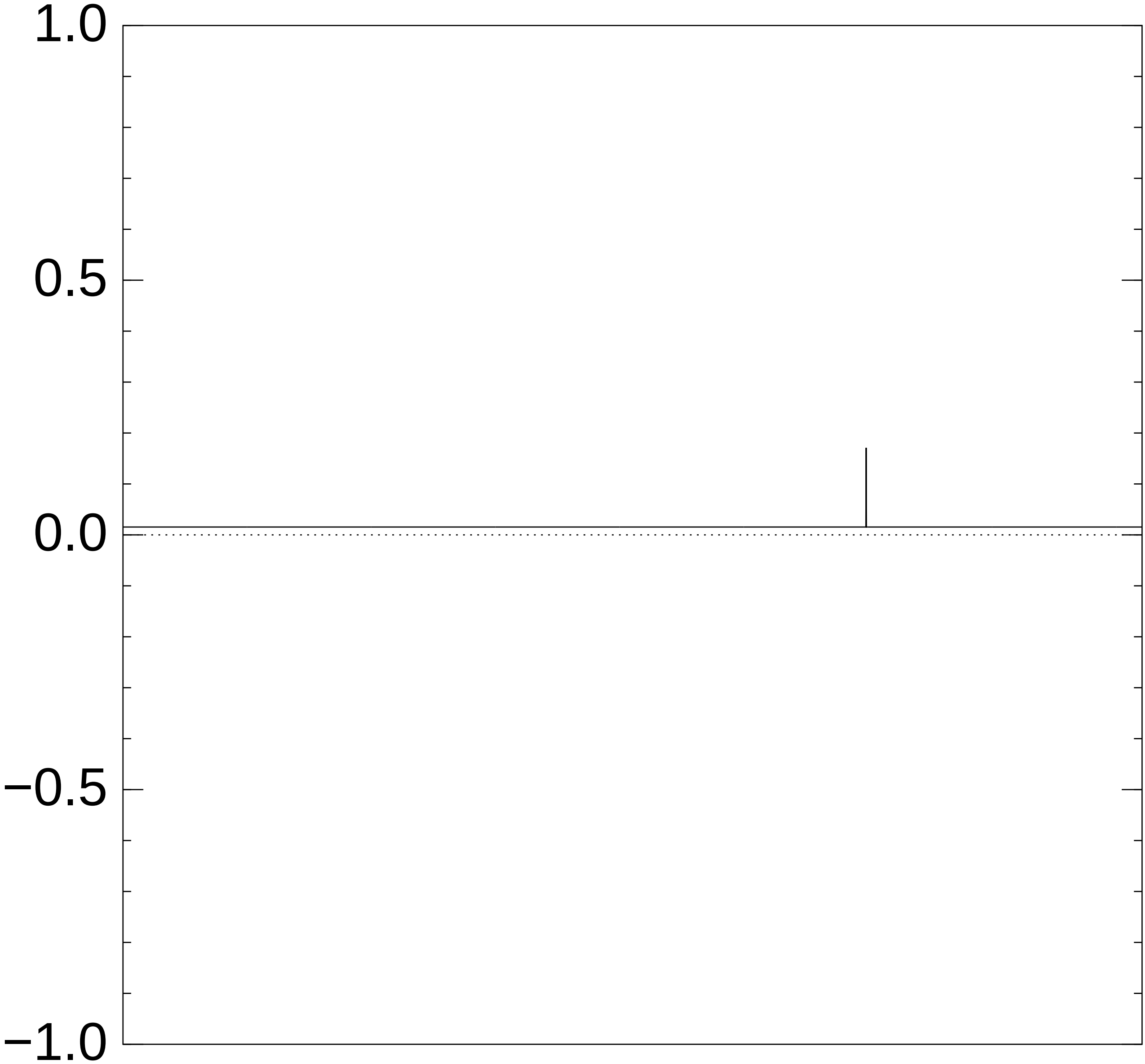
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $3 \times$ (Step $1 +$ Step 2):



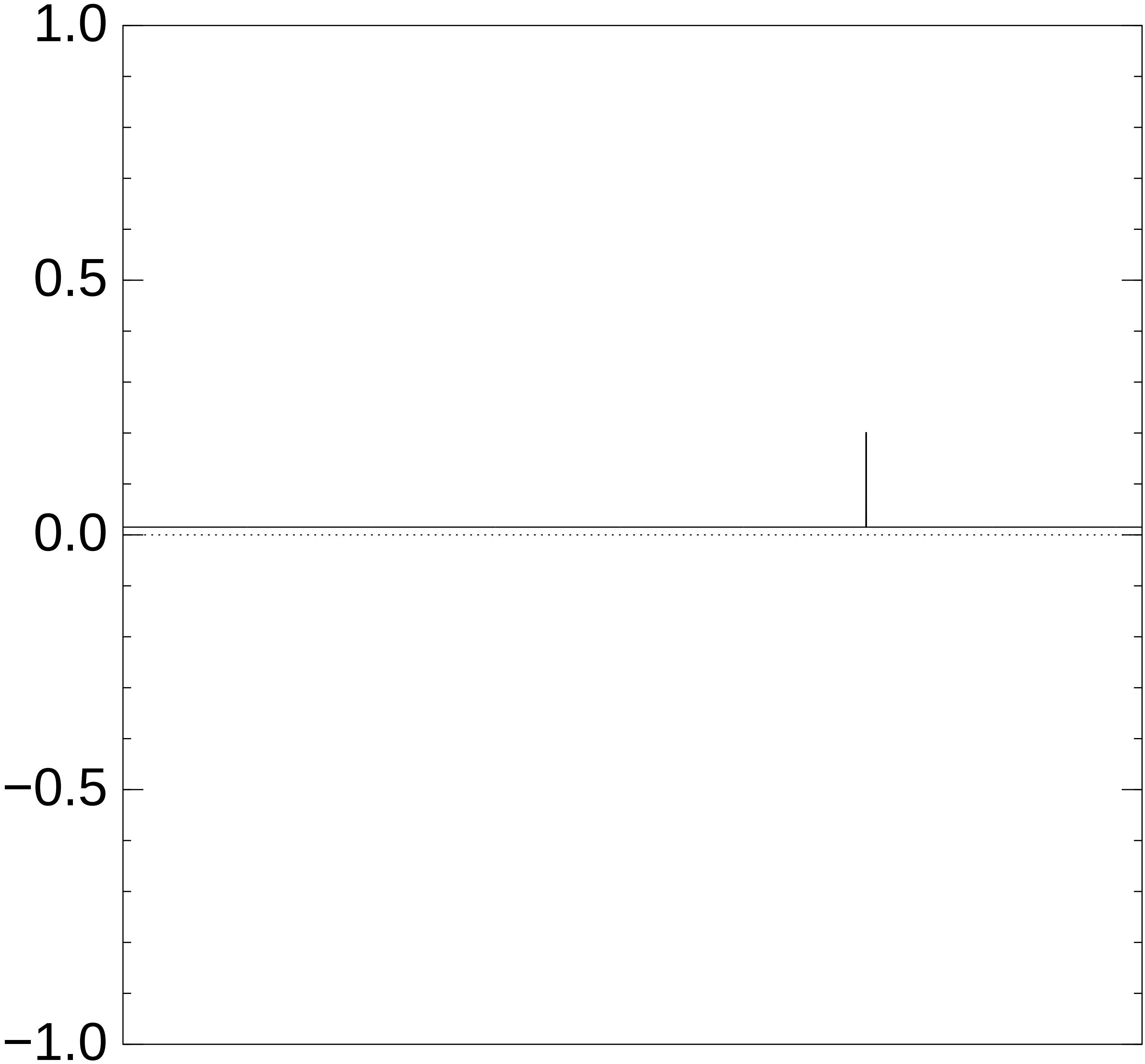
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $4 \times$ (Step 1 + Step 2):



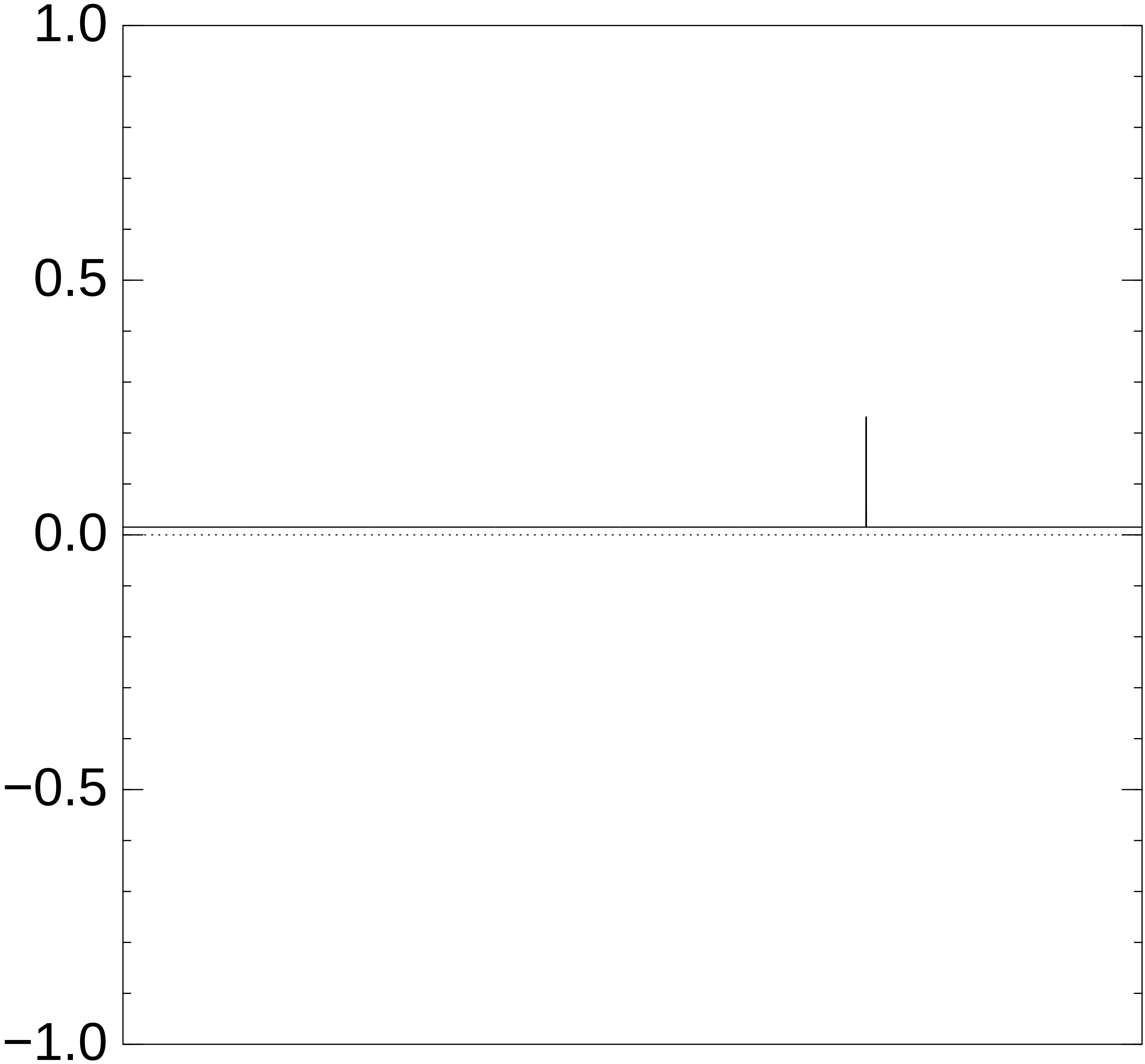
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $5 \times (\text{Step } 1 + \text{Step } 2)$:



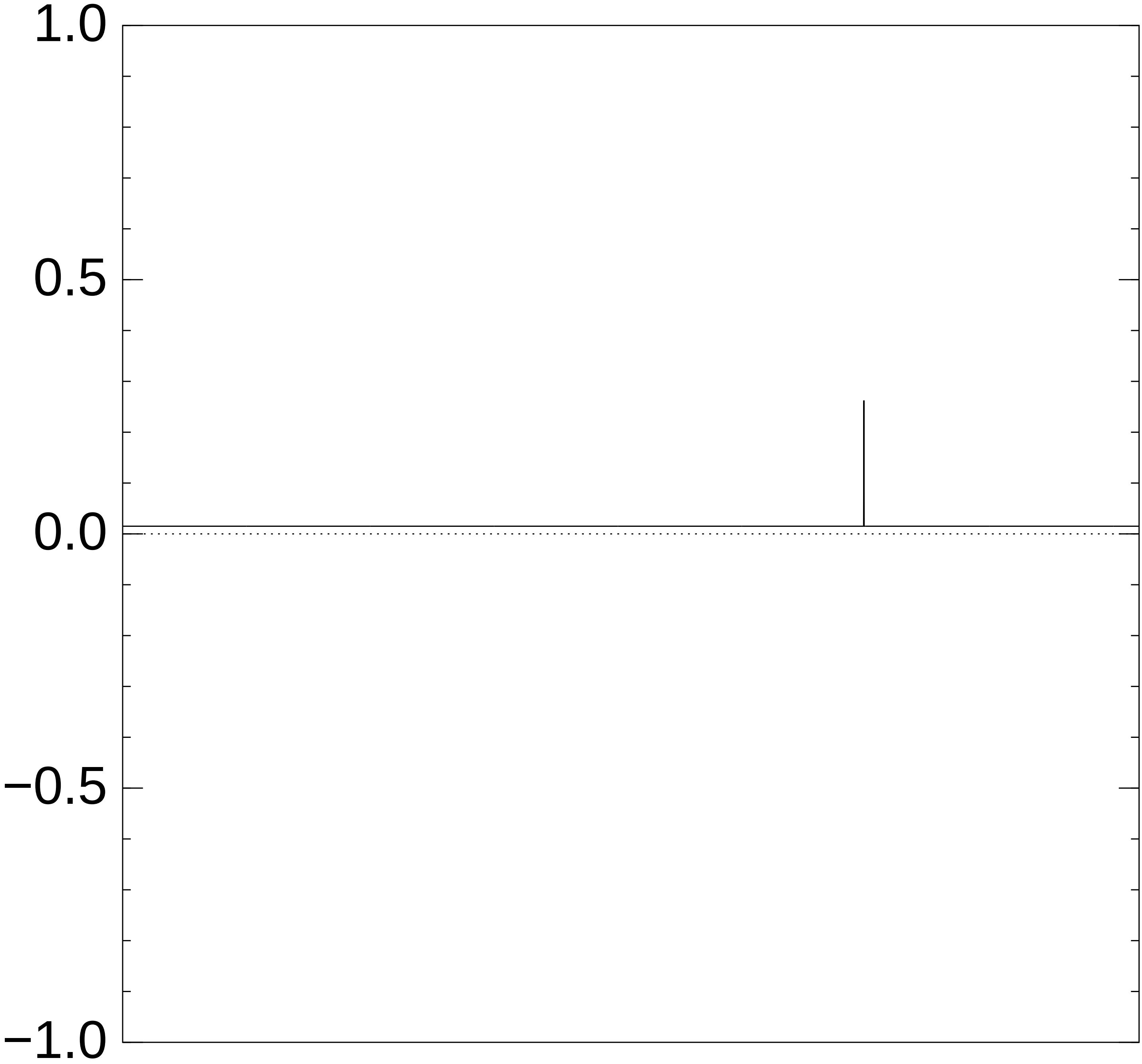
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $6 \times (\text{Step 1} + \text{Step 2})$:



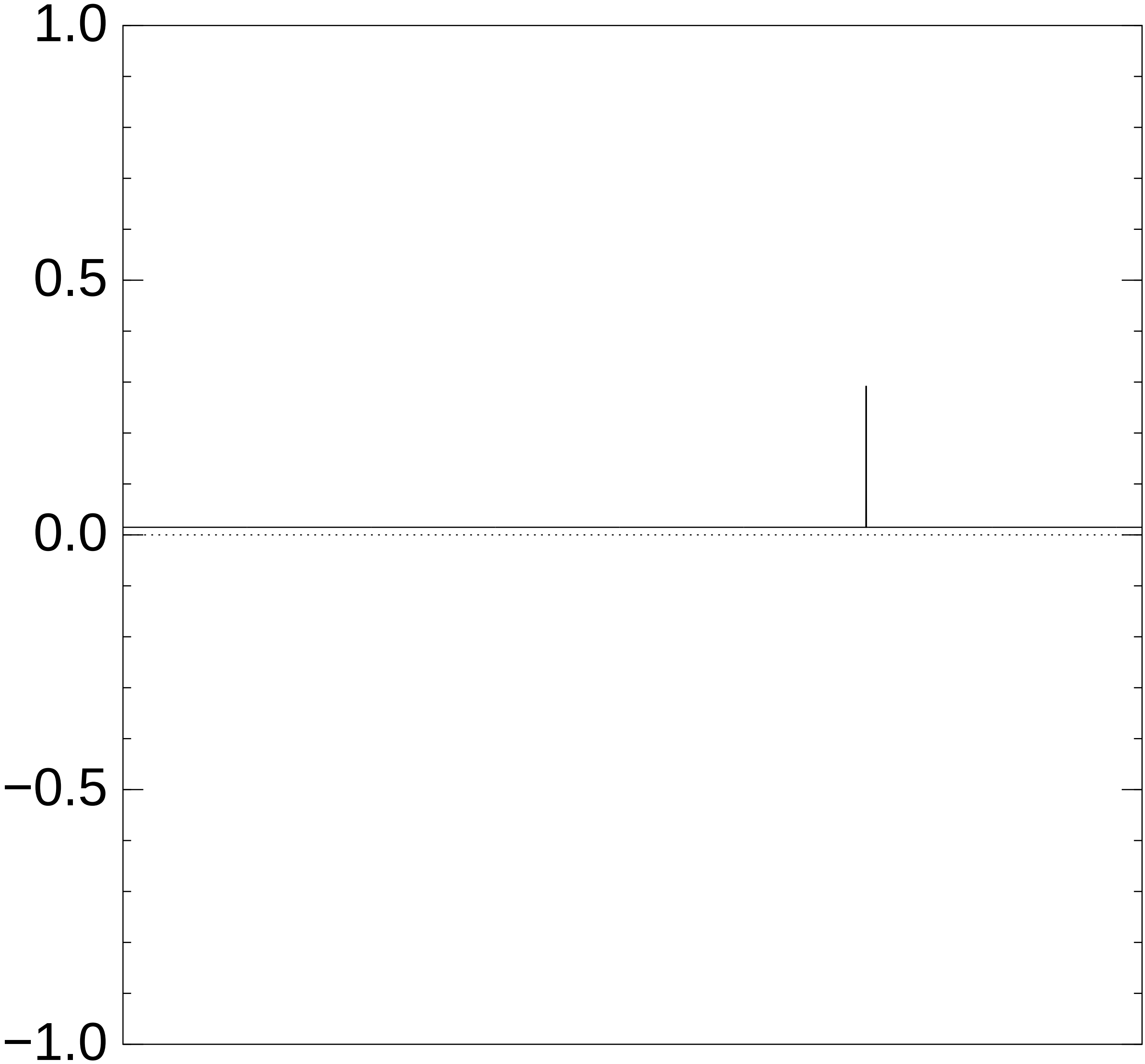
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $7 \times$ (Step 1 + Step 2):



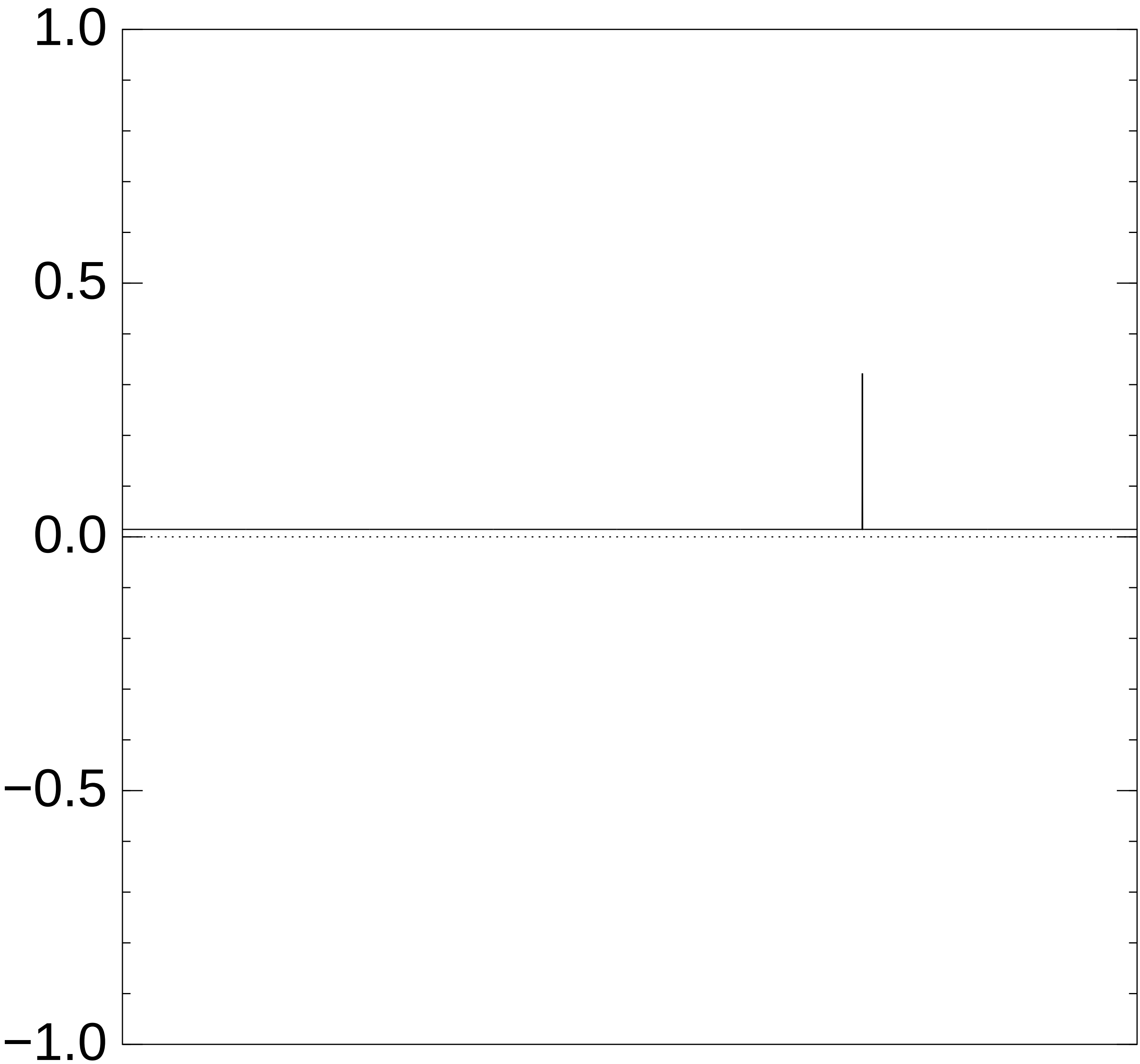
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $8 \times (\text{Step 1} + \text{Step 2})$:



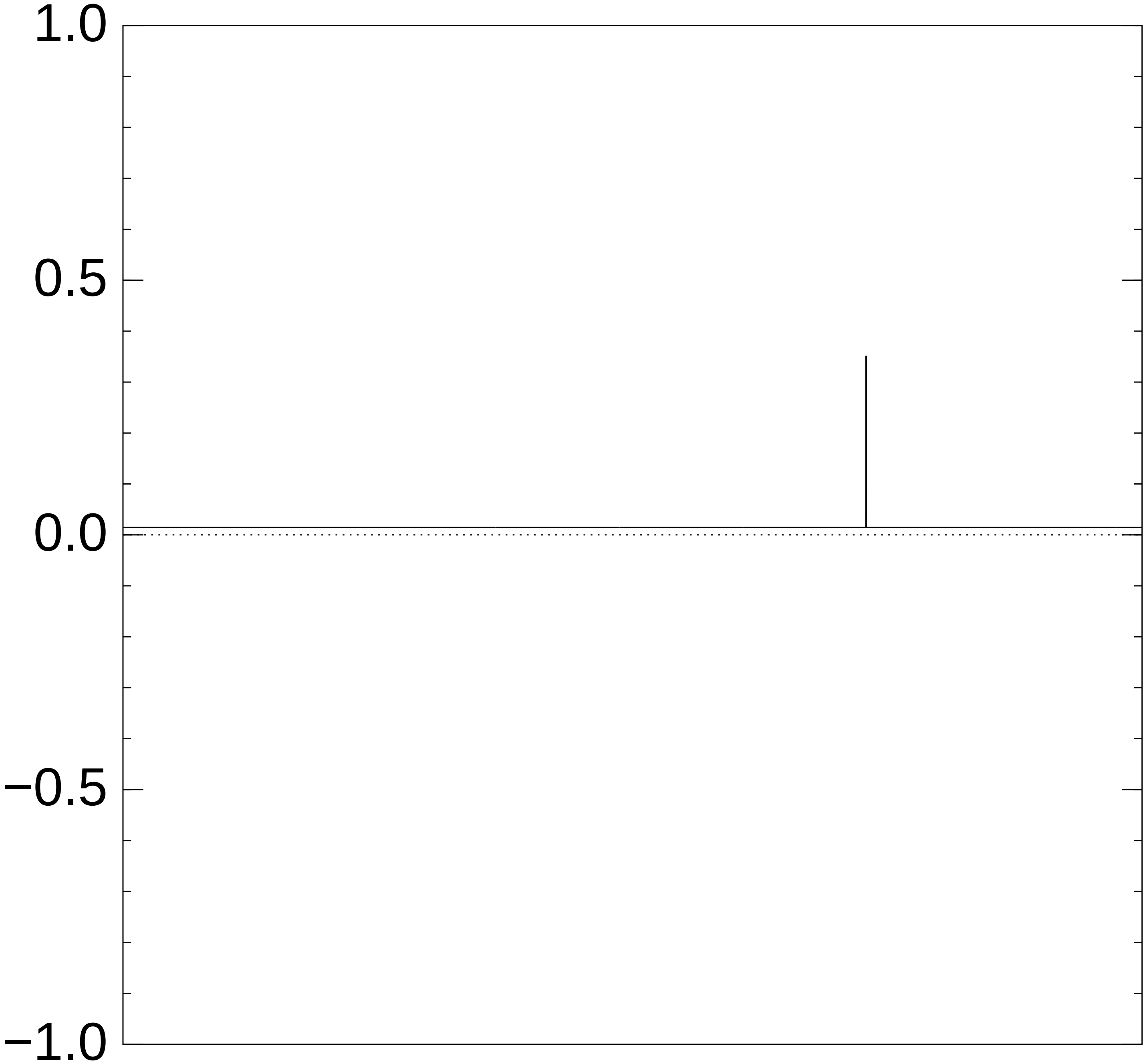
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $9 \times (\text{Step 1} + \text{Step 2})$:



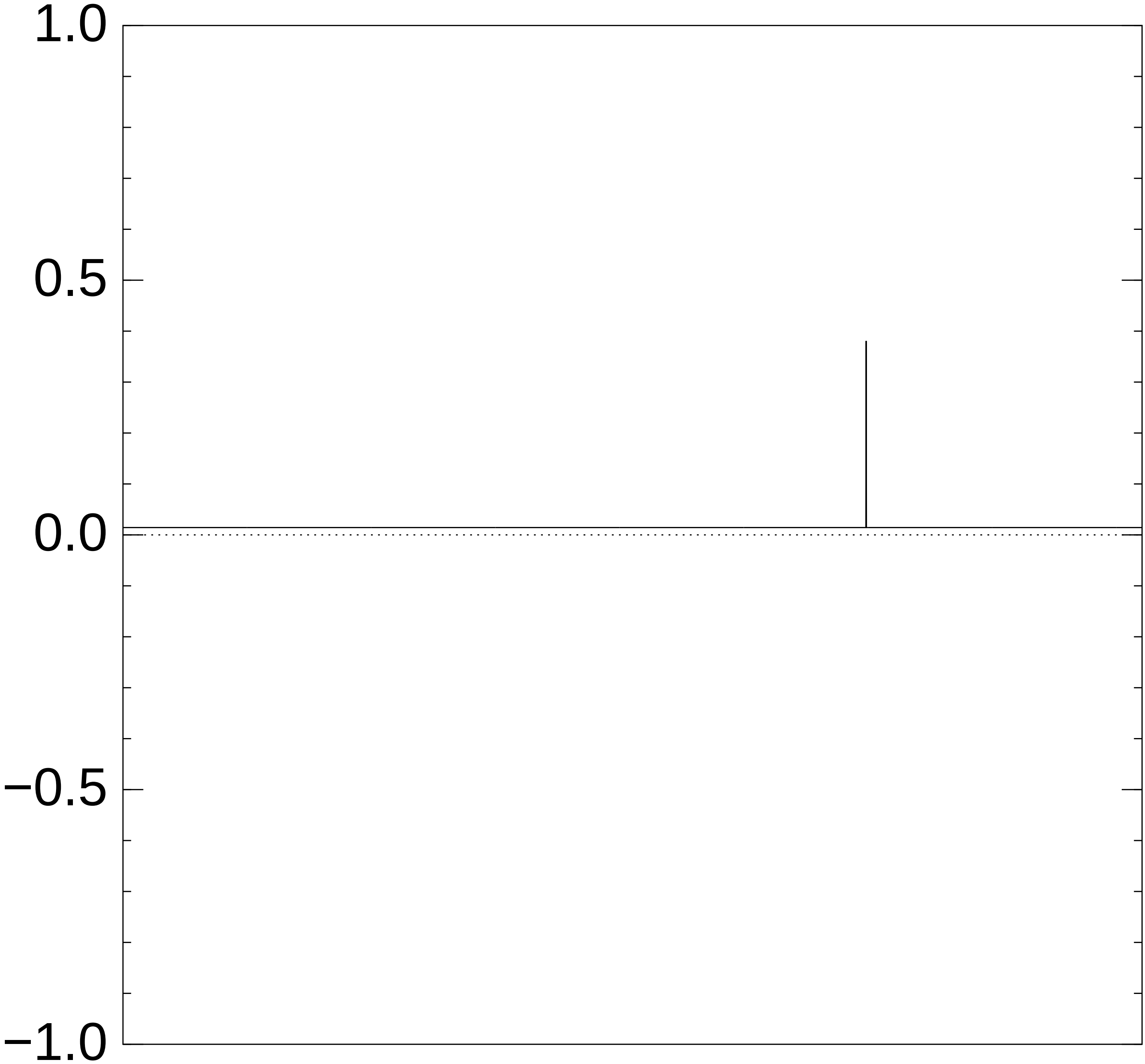
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $10 \times (\text{Step } 1 + \text{Step } 2)$:



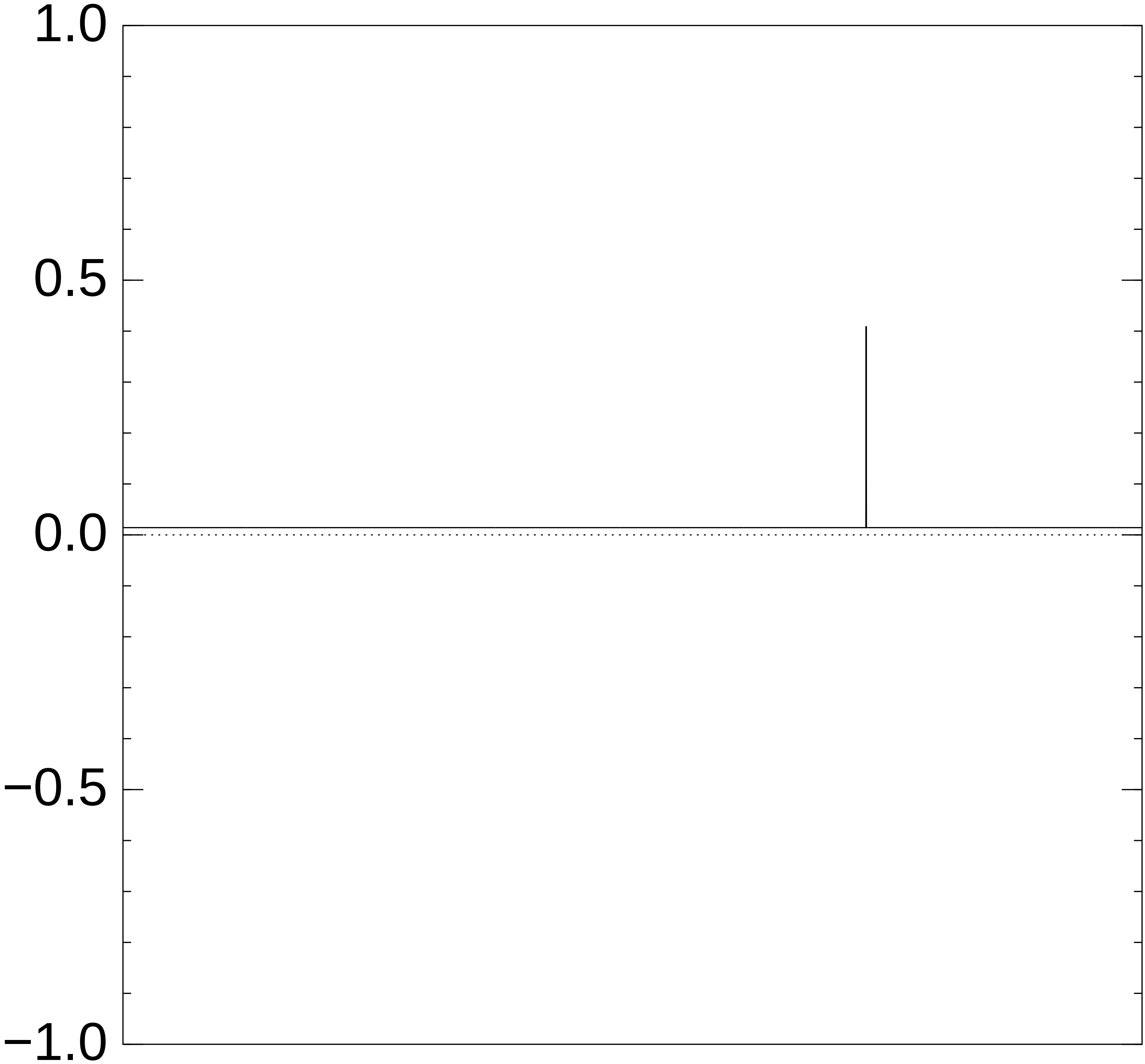
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $11 \times (\text{Step 1} + \text{Step 2})$:



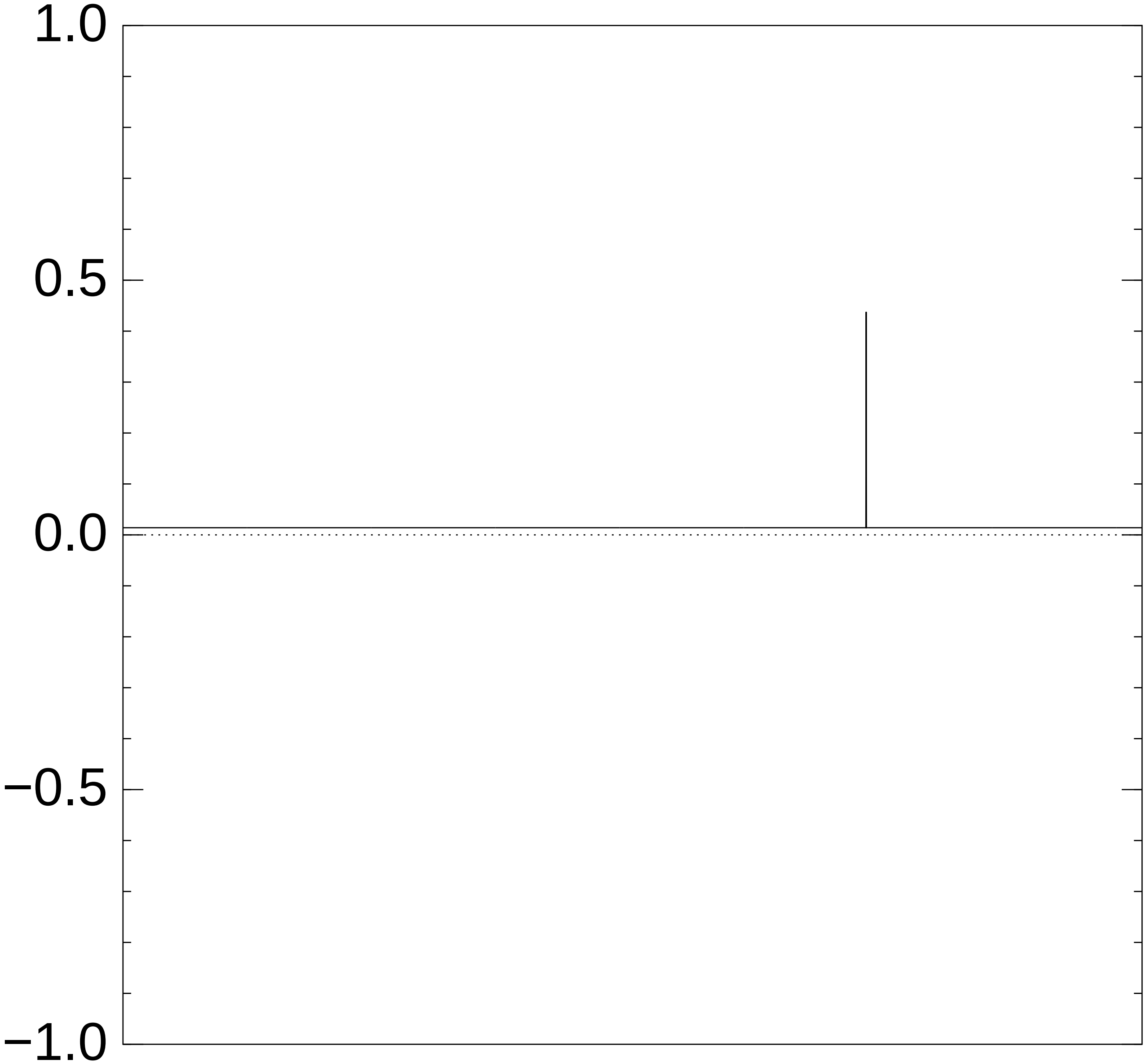
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $12 \times$ (Step $1 +$ Step 2):



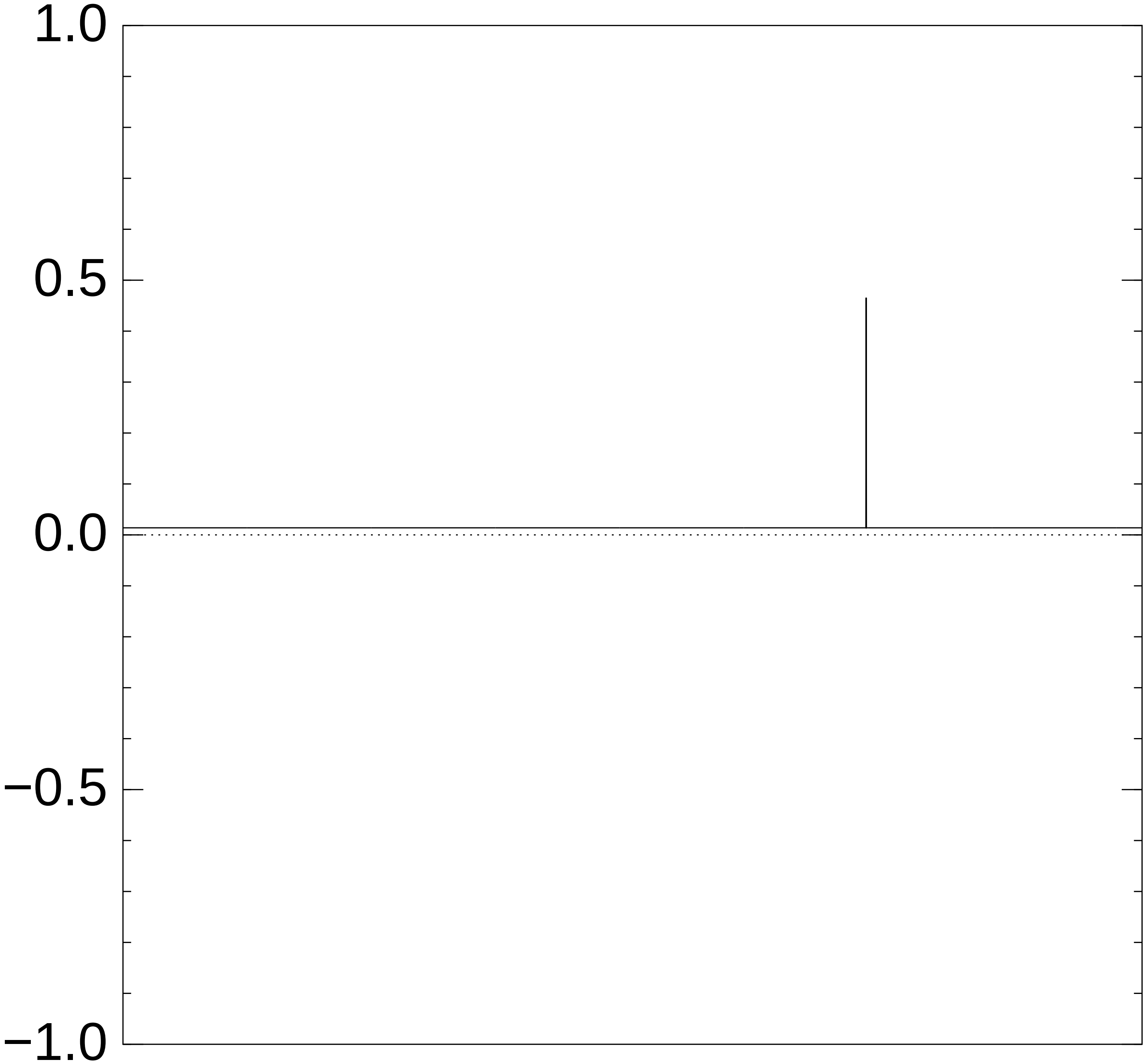
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $13 \times (\text{Step 1} + \text{Step 2})$:



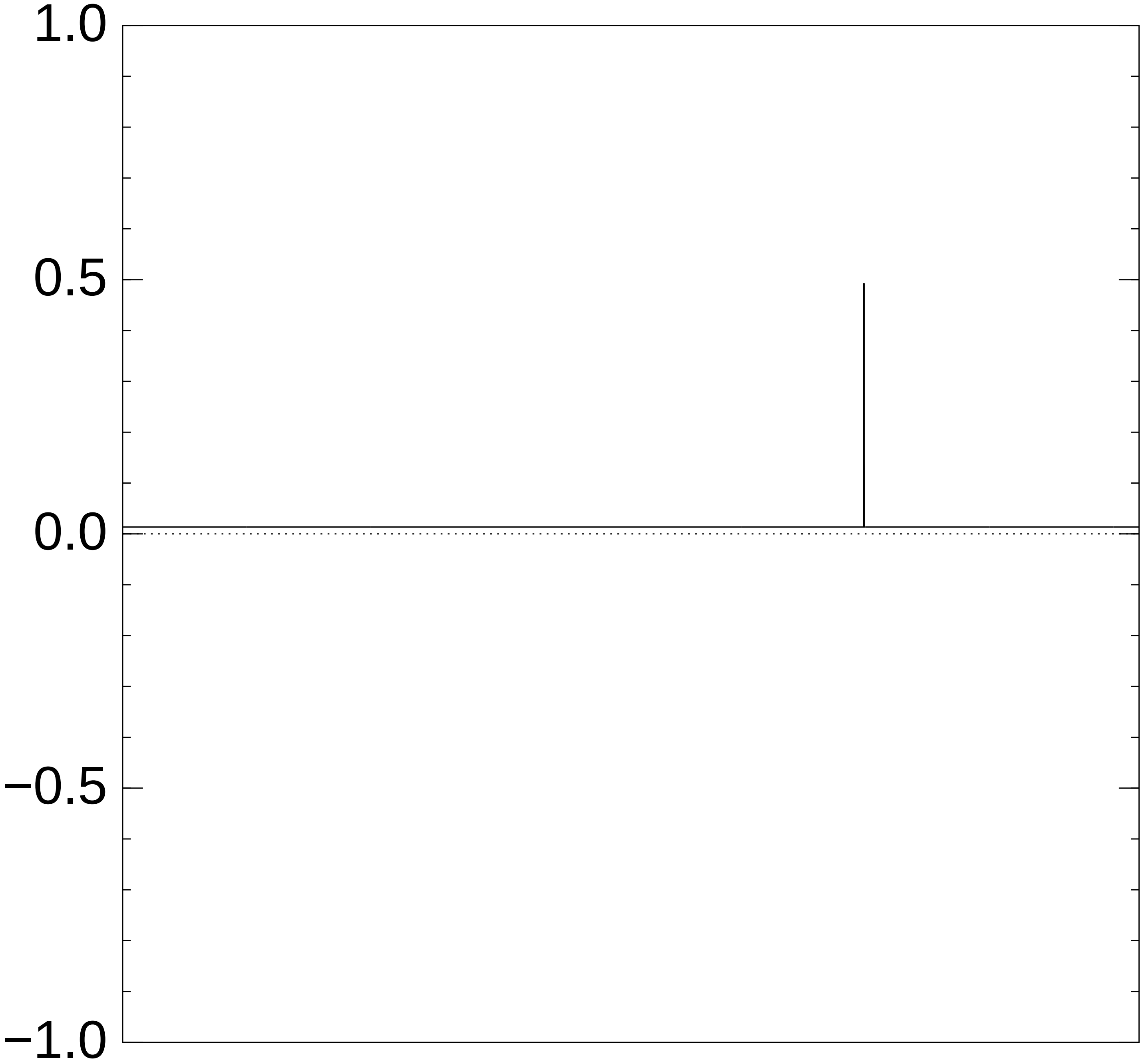
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $14 \times$ (Step 1 + Step 2):



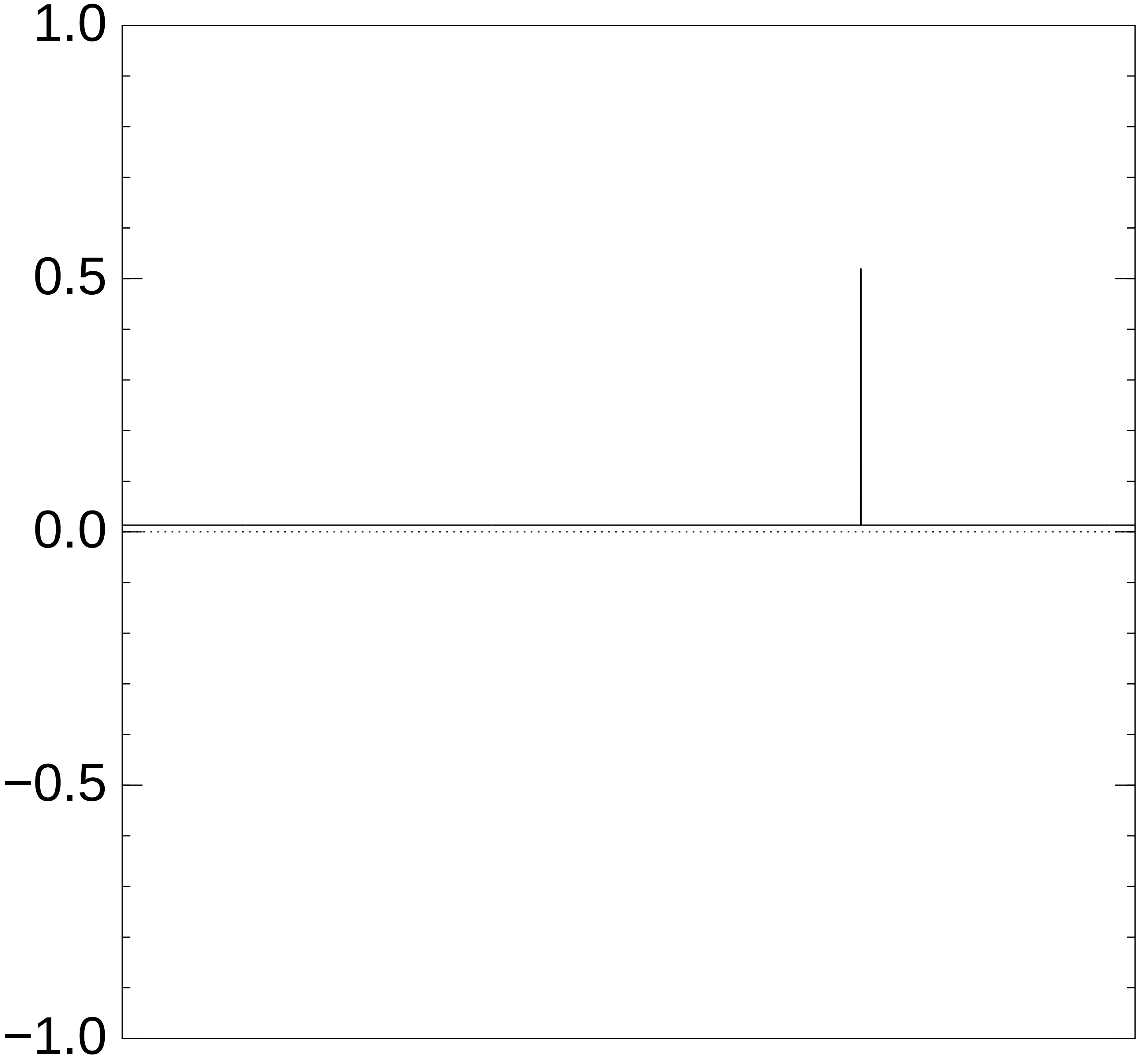
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $15 \times (\text{Step 1} + \text{Step 2})$:



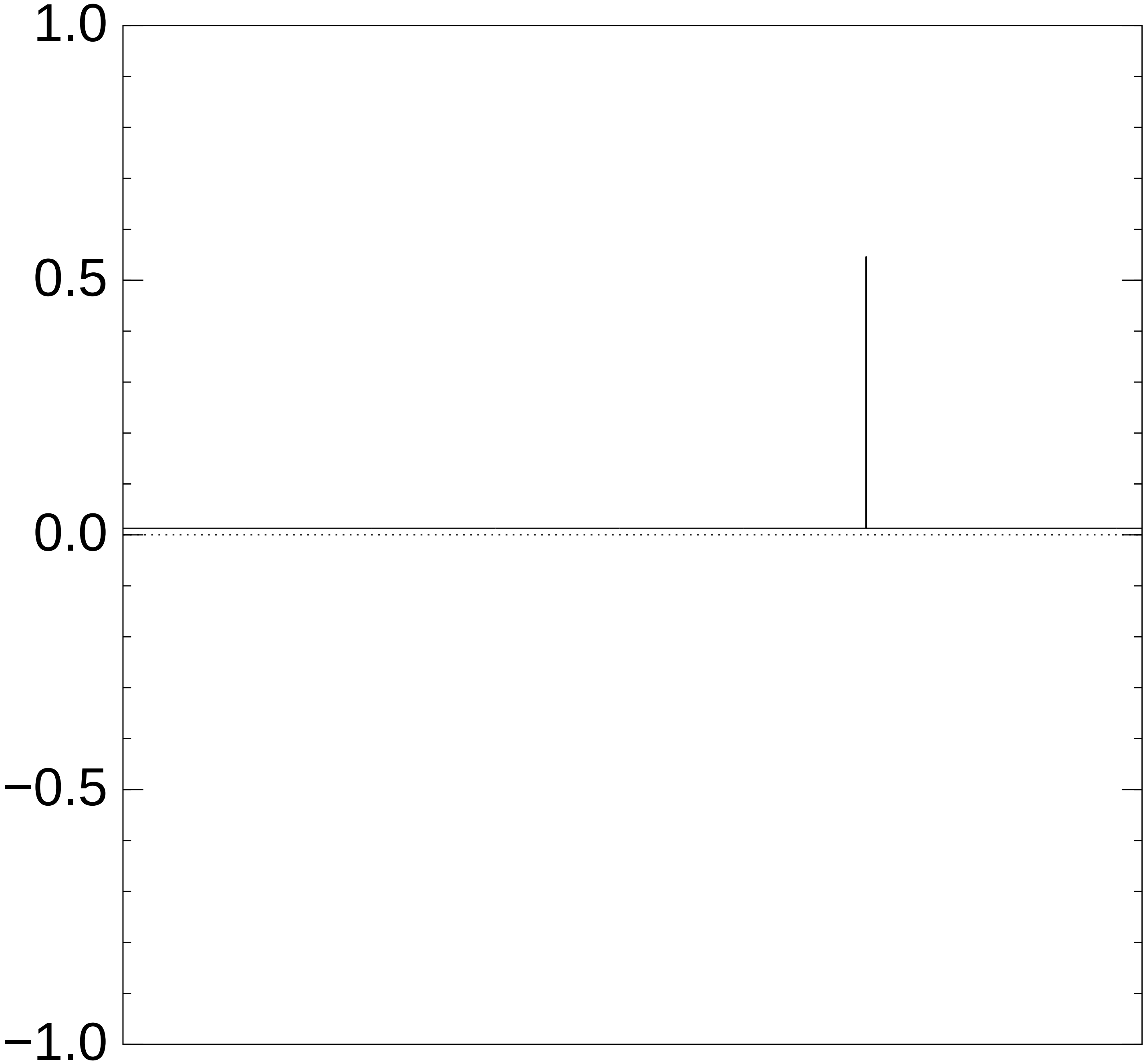
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $16 \times (\text{Step } 1 + \text{Step } 2)$:



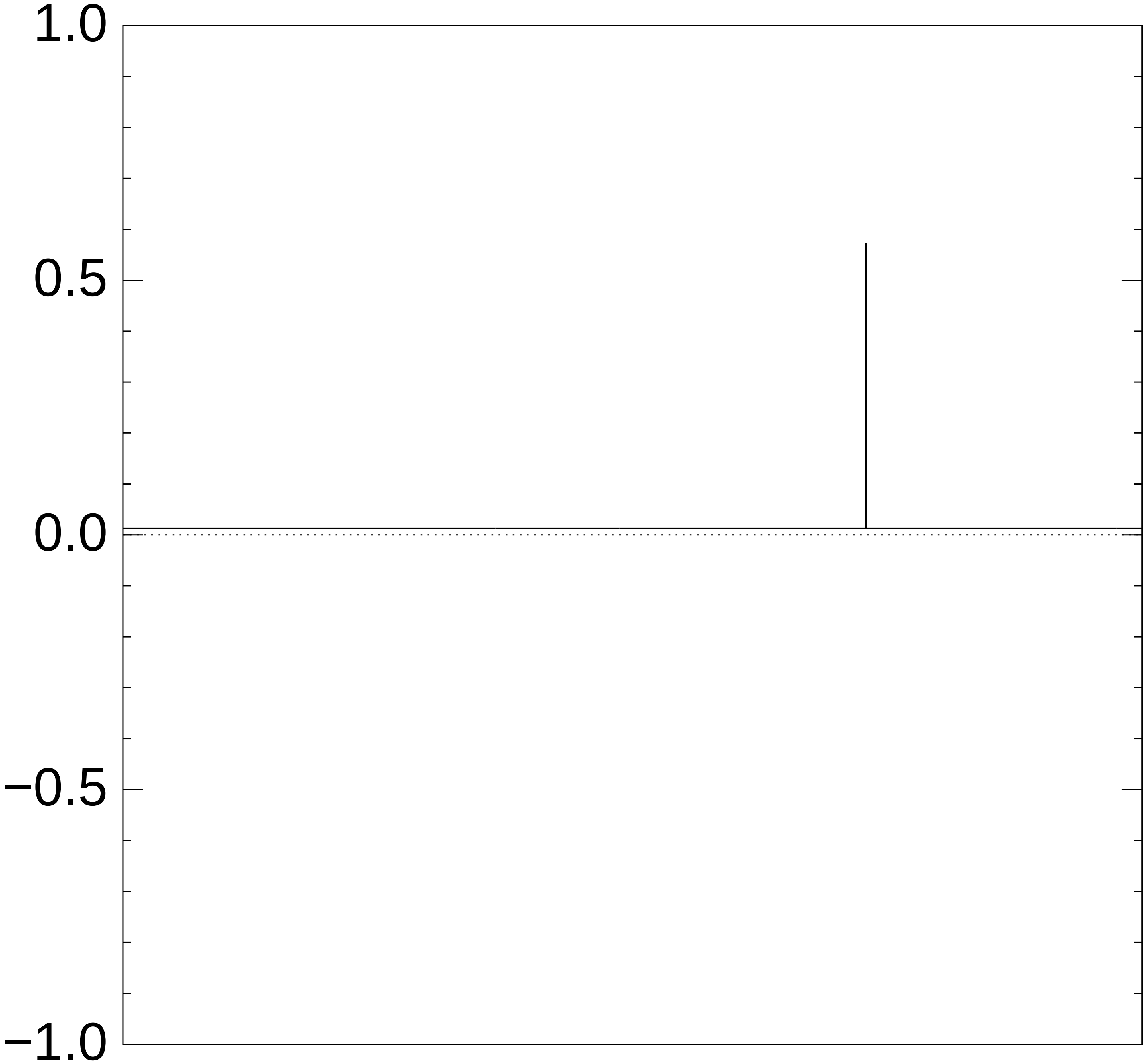
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $17 \times$ (Step $1 +$ Step 2):



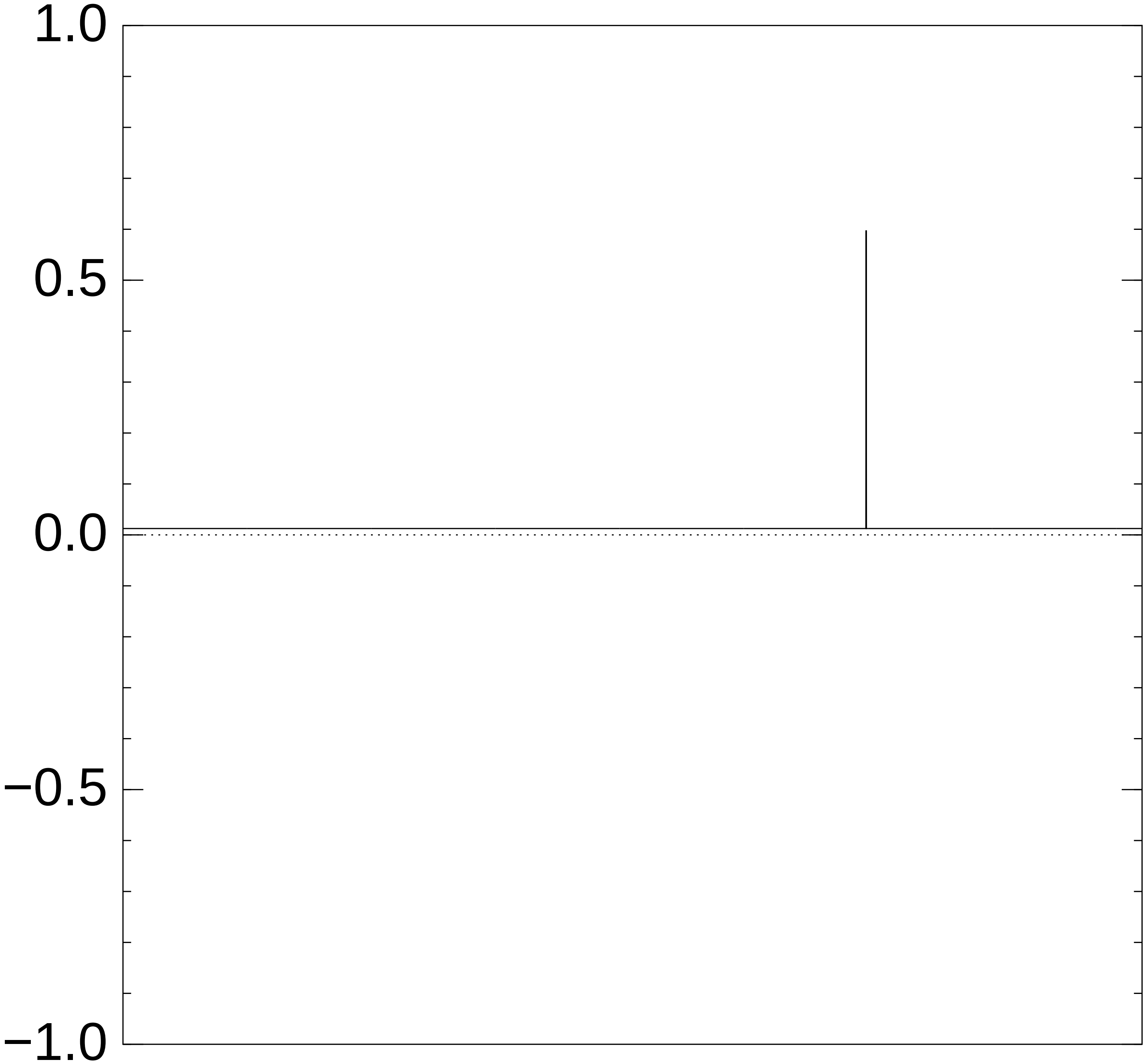
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $18 \times (\text{Step } 1 + \text{Step } 2)$:



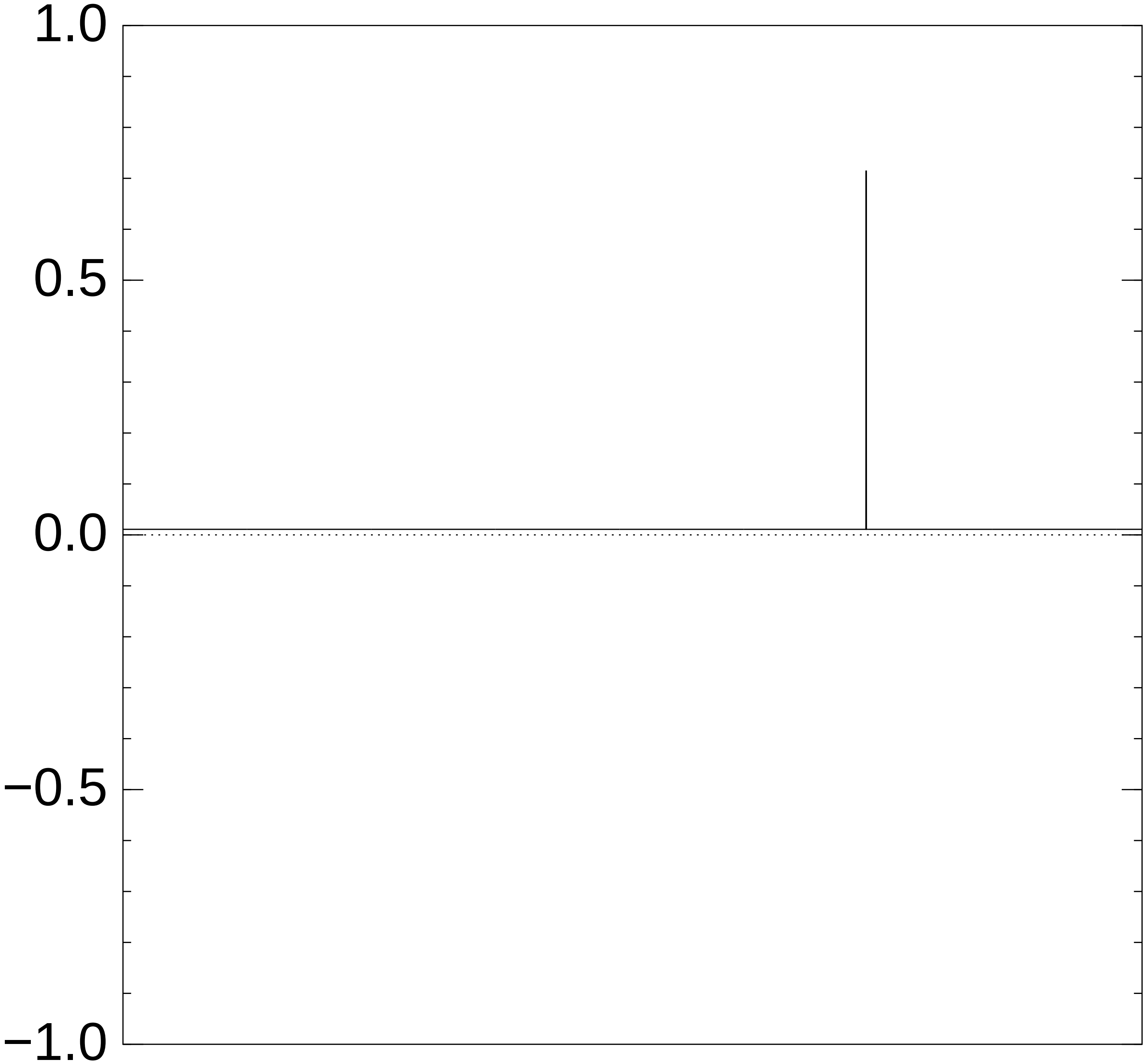
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $19 \times$ (Step 1 + Step 2):



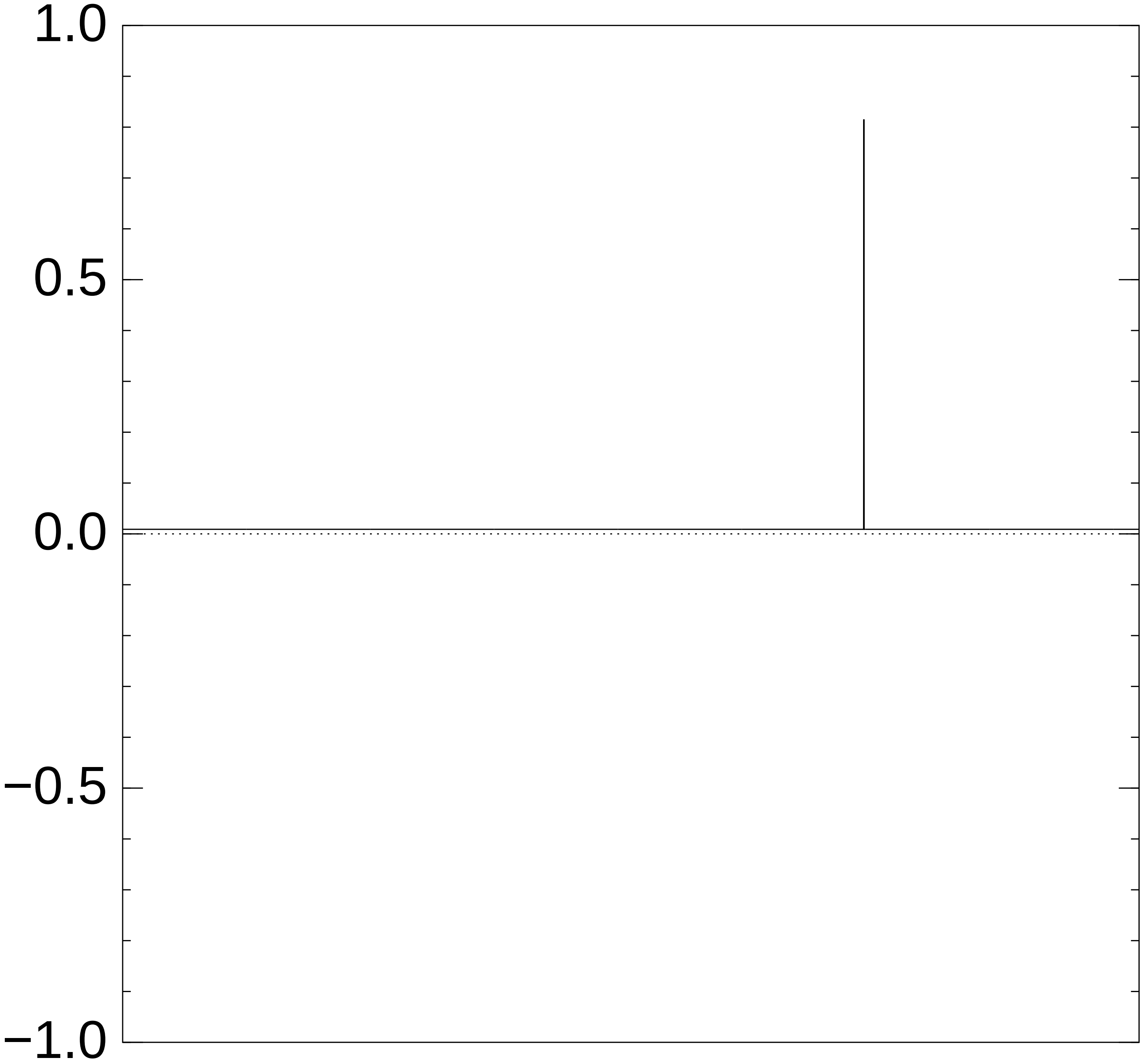
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $20 \times$ (Step $1 +$ Step 2):



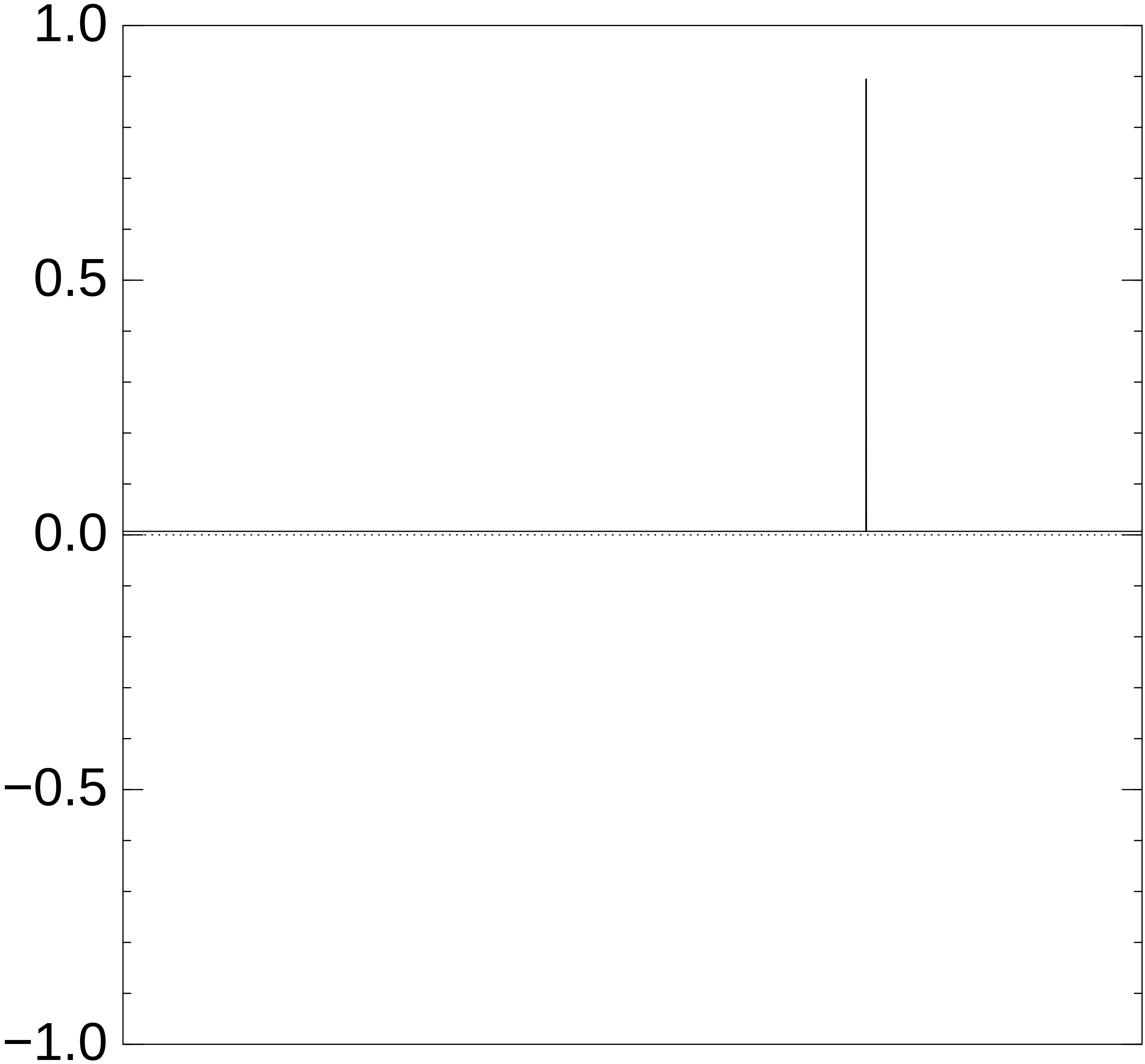
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $25 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $30 \times$ (Step 1 + Step 2):

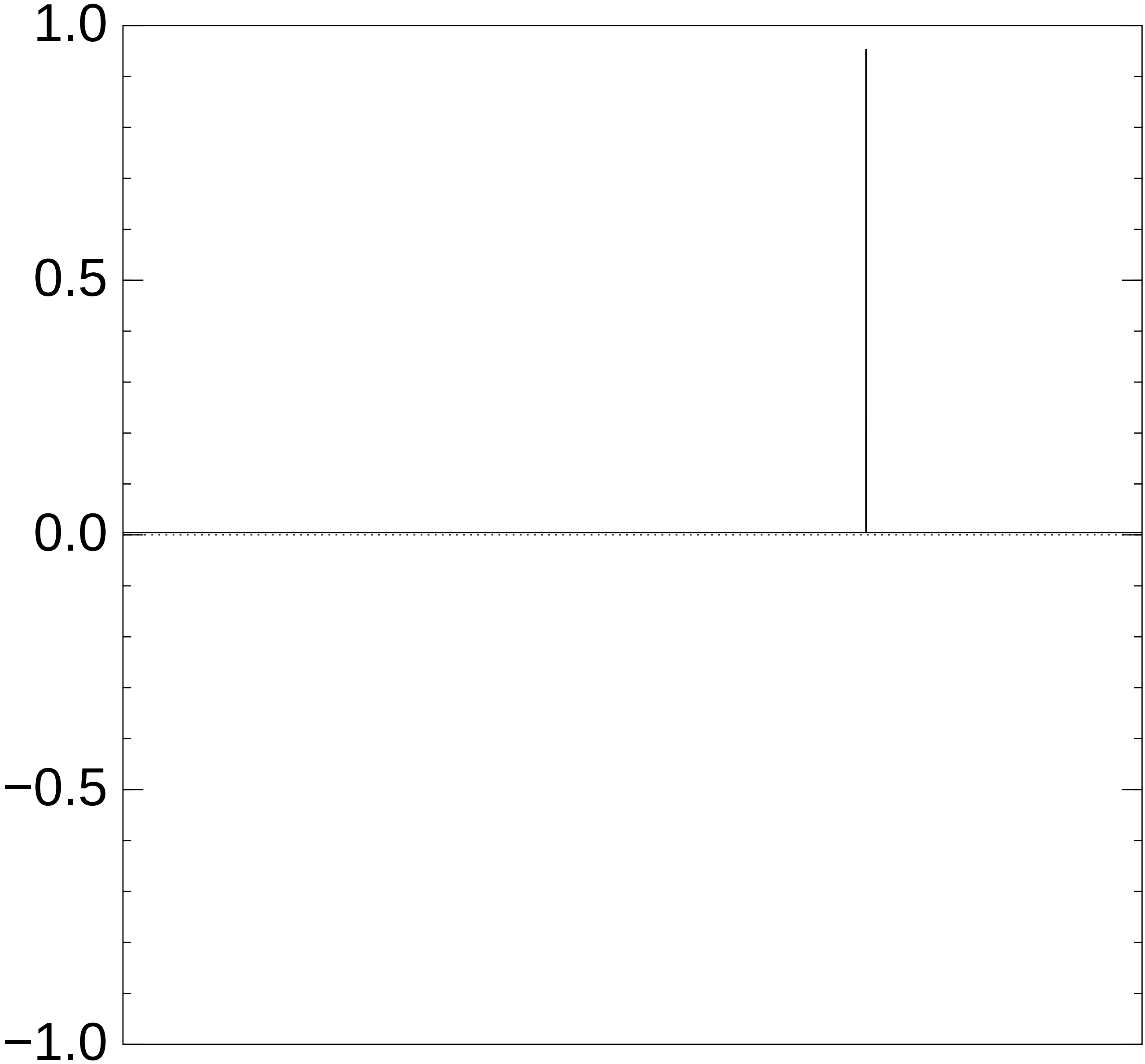


Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $35 \times (\text{Step } 1 + \text{Step } 2)$:

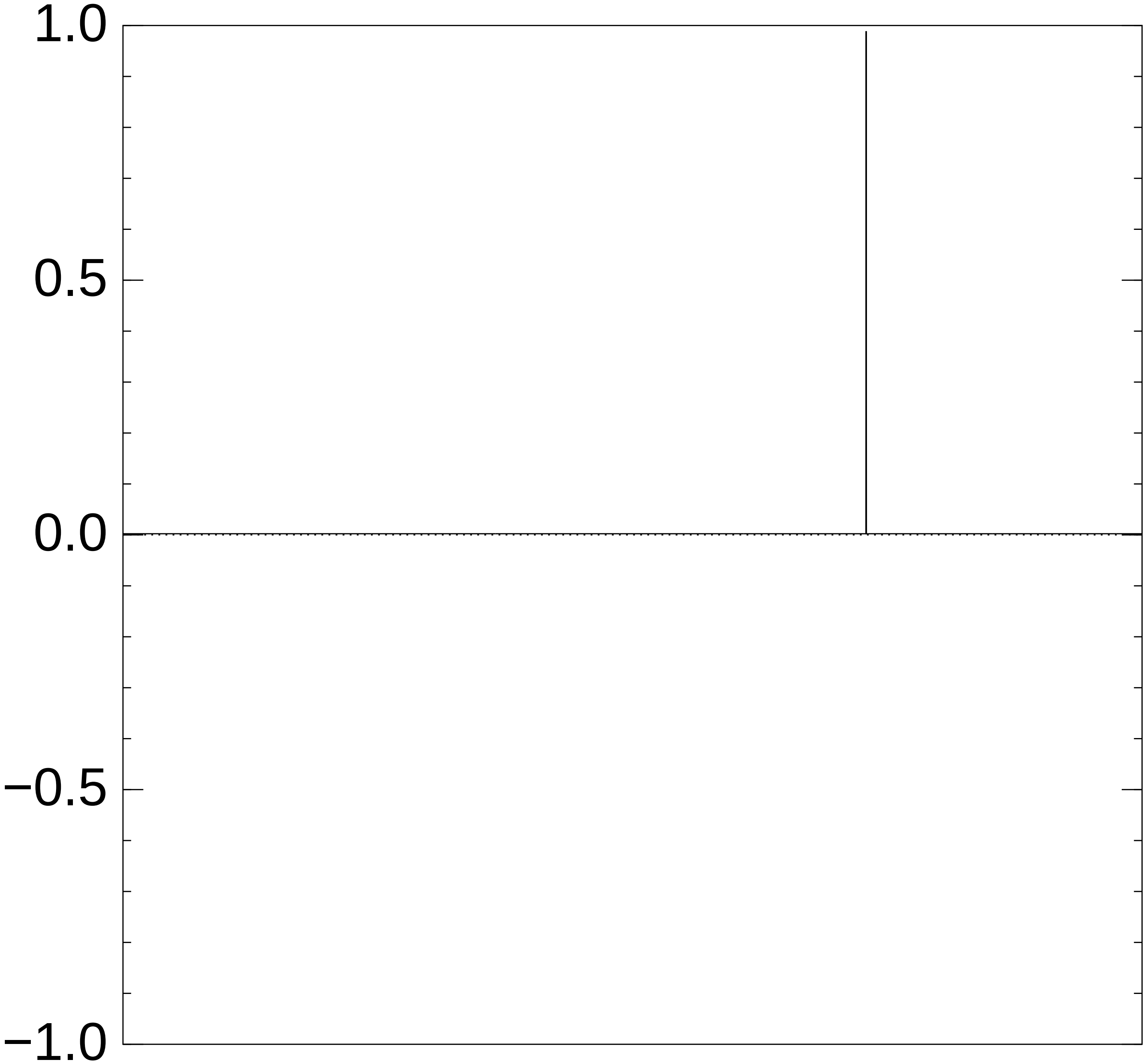


Good moment to stop, measure.

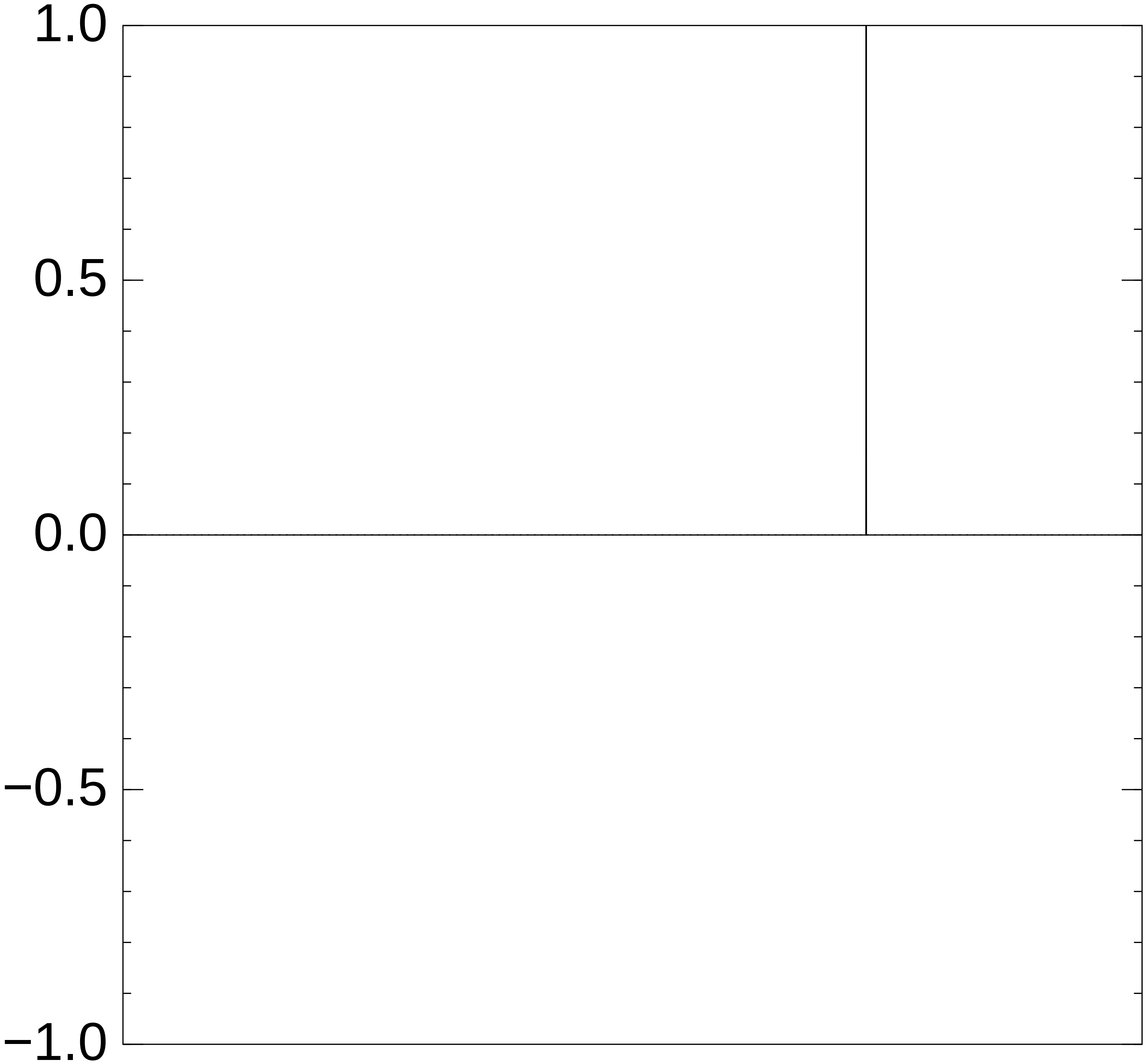
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $40 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $45 \times (\text{Step } 1 + \text{Step } 2)$:

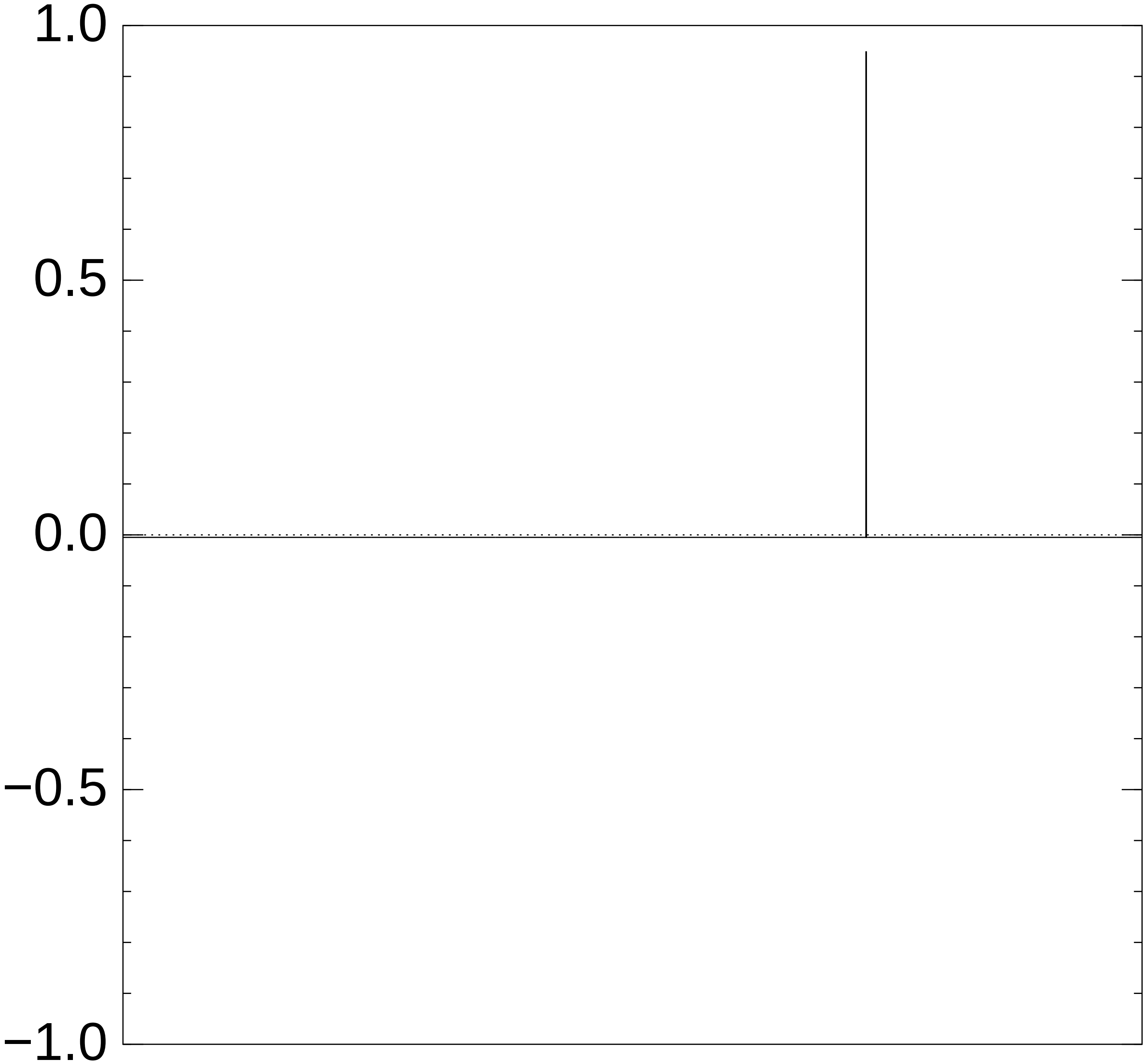


Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $50 \times$ (Step $1 +$ Step 2):

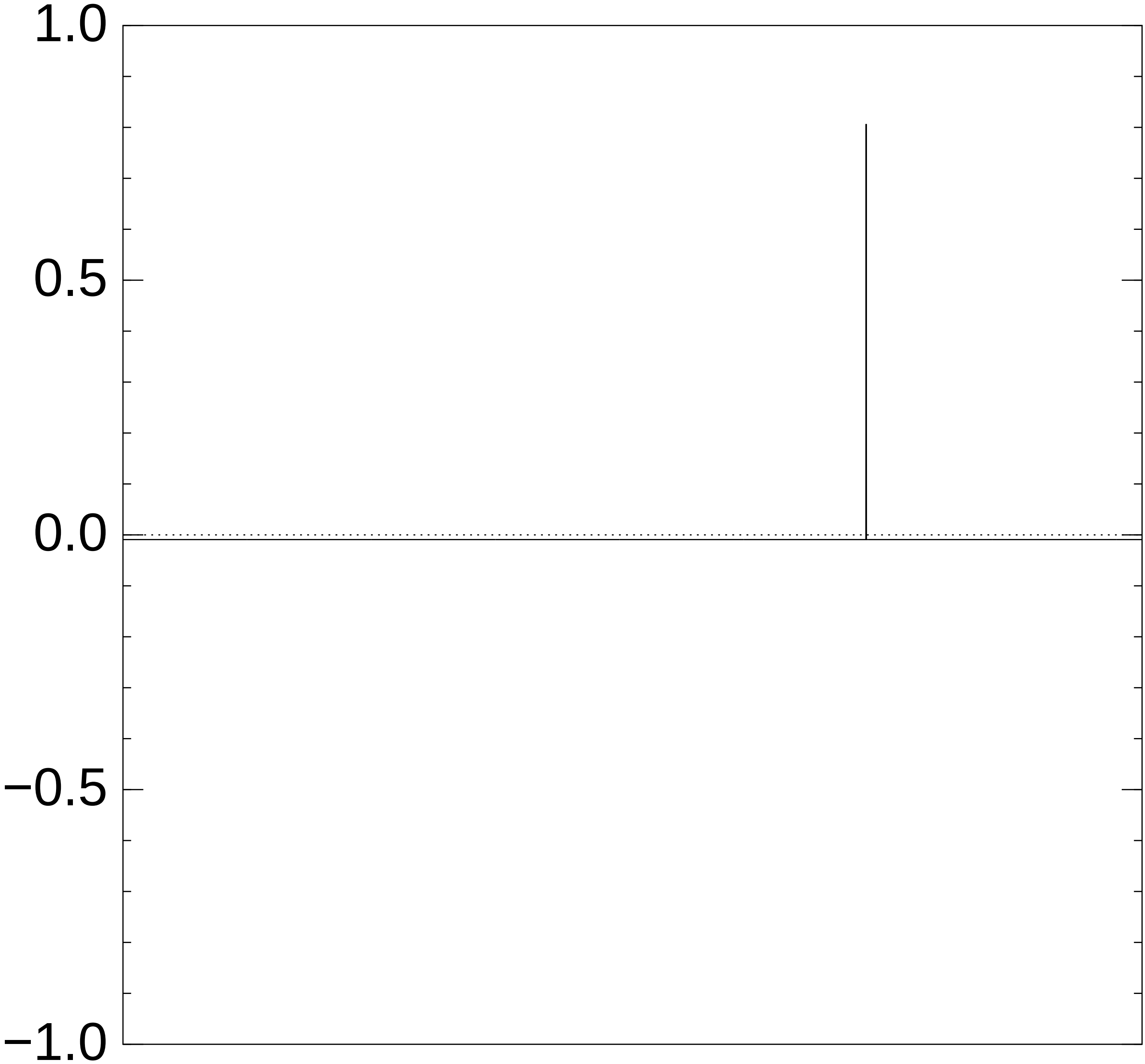


Traditional stopping point.

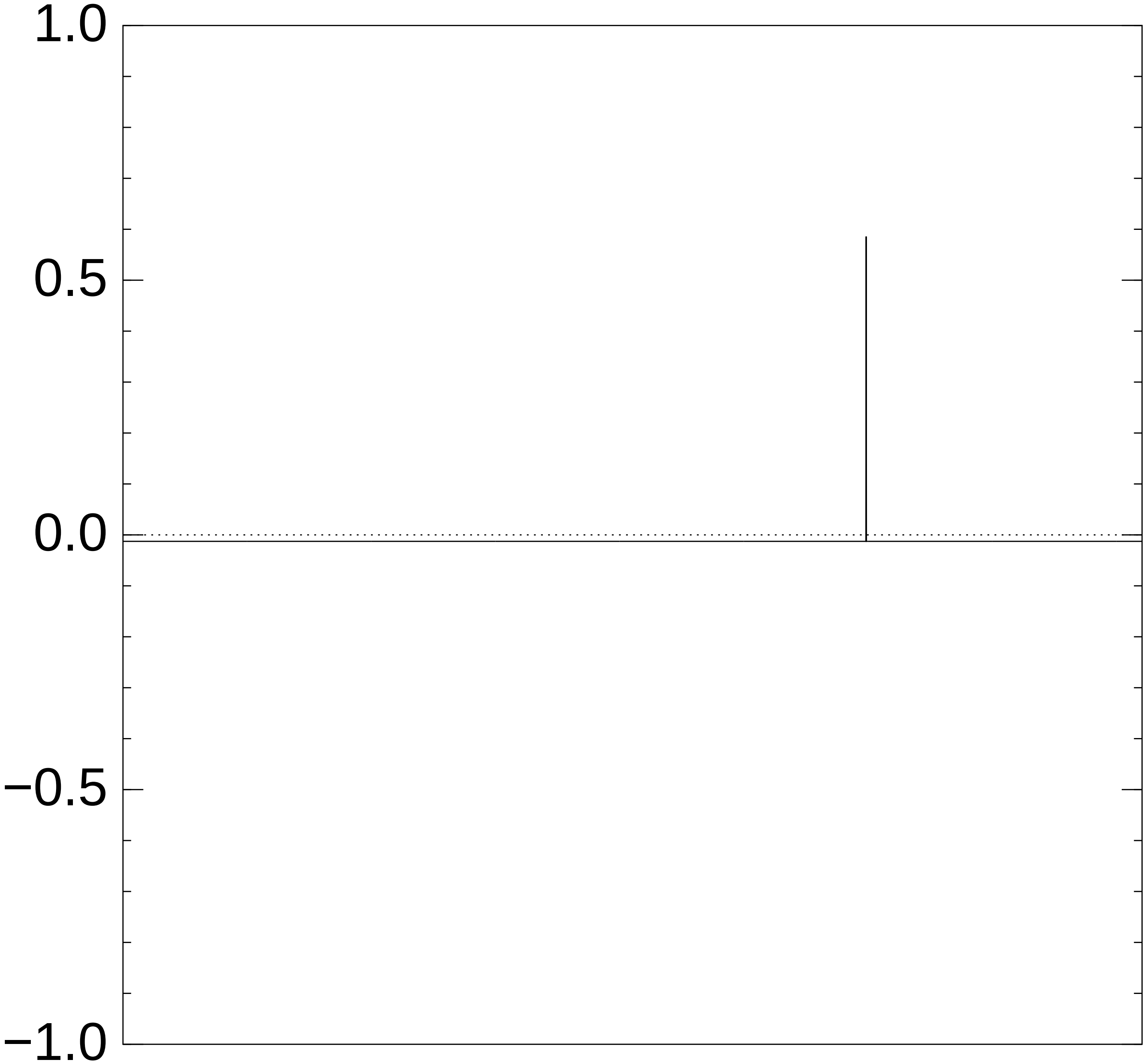
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $60 \times$ (Step $1 +$ Step 2):



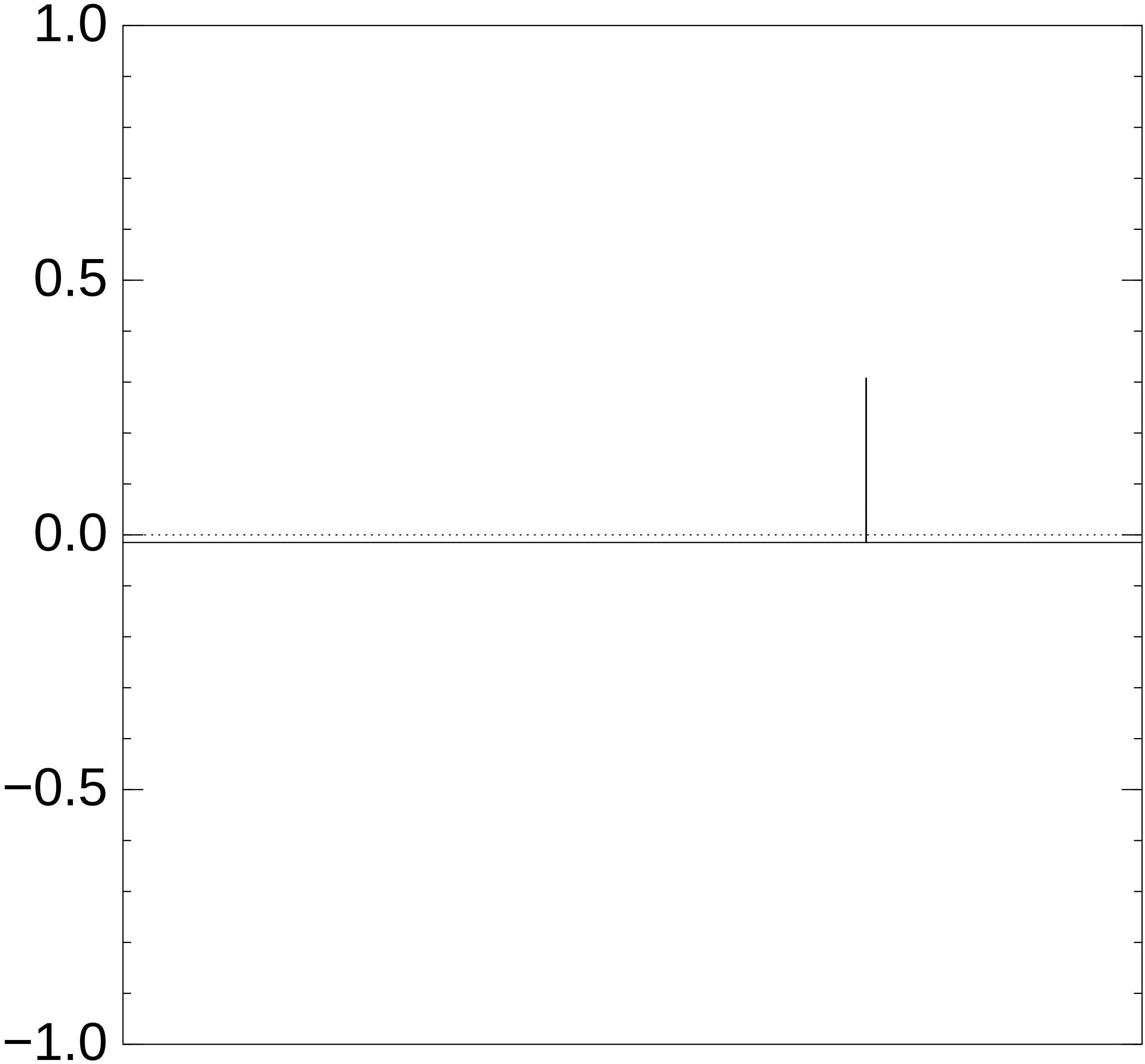
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $70 \times$ (Step $1 +$ Step 2):



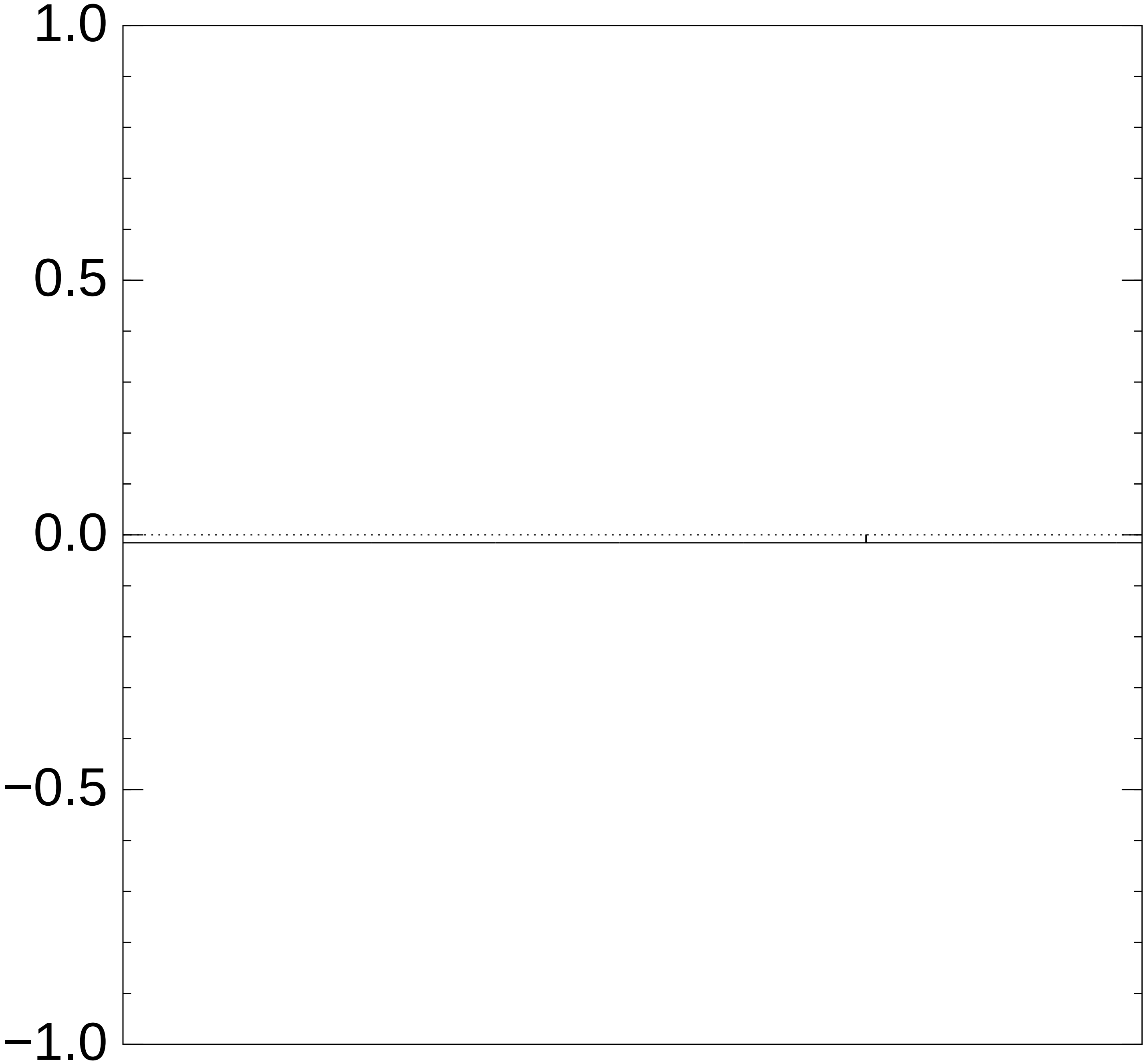
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $80 \times$ (Step 1 + Step 2):



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $90 \times (\text{Step 1} + \text{Step 2})$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $100 \times$ (Step 1 + Step 2):



Very bad stopping point.

$q \mapsto a_q$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) a_q for roots q;
(2) a_q for non-roots q.

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Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is ≈ 1
after $\approx (\pi/4) 2^{0.5n}$ iterations.

Many more applications

Shor generalizations:
e.g., poly-time attack breaking "cyclotomic" case of Gentry STOC 2009 "Fully homomorphic encryption using ideal lattices".

Grover generalizations:
e.g., fastest subset-sum attacks use "quantum walks".

Not just Shor and Grover:
e.g., subexponential-time CRS/CSIDH isogeny attack uses "Kuperberg's algorithm".